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COHOMOLOGIES AND DEFORMATIONS OF RIGHT-SYMMETRIC ALGEBRAS

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An algebra A with identity $(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b)$, is called right-symmetric. The cohomology and deformation theories for right-symmetric algebras are developed. The cohomology of gl_n and half-Witt algebras W_n^{rsym}, $p = 0$, $W_n^{rsym}(\mathrm{m})$, $p > 0$, are calculated. In particular, one right-symmetric central extension of W_1^{rsym} is constructed.

1. Introduction

An algebra A over a field $\mathcal{K}$ of characteristic $p \geq 0$ is called right-symmetric [26], [19] if for any $a, b, c \in A$, the following condition takes place:

$$a \circ (b \circ c) - (a \circ b) \circ c = a \circ (c \circ b) - (a \circ c) \circ b.$$

Any associative algebra is right-symmetric. For example, gl_n under the usual multiplication of matrices is right-symmetric. An algebra of vector fields $\mathcal{K}[[x^{\pm 1}, \ldots, x_n^{\pm 1}]]$ under multiplication $u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i$ gives us a less trivial example of right-symmetric algebras. It is not associative. Since its Lie algebra is isomorphic to the Witt algebra W_n, we call it a half-Witt algebra and denote it by W_n^{rsym}. If $n = 1$, this algebra satisfies one more identity:

$$a \circ (b \circ c) = b \circ (a \circ c).$$

Such algebras are called Novikov algebras [1], [21]. The generalization of the Novikov structure for the case $n > 1$ is possible if we consider the half-Witt algebra not with one, but with two multiplications. If we endow $\mathcal{K}[[x^{\pm 1}, \ldots, x^{\pm 1}]]$ with the second multiplication $u\partial_i * v\partial_j = \partial_i(u)v\partial_j$, then we obtain an algebra with the following identities:

$$a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b = 0,$$

$$a * (b * c) - b * (a * c) = 0,$$

$$a \circ (b * c) - b * (a \circ c) = 0,$$

$$(a * b - b * a - a \circ b + b \circ a) * c = 0,$$

$$(a \circ b - b \circ a) * c + a * (c \circ b) - (a * c) \circ b - b * (c \circ a) + (b * c) \circ a = 0.$$

These two multiplications are useful in constructing the right-symmetric, Chevalley–Eilenberg, and Leibniz cocycles of such algebras.

We develop the cohomology theory for right-symmetric algebras. We endow the right-symmetric cochain complex $C^*_{rsym}(A, M) = \oplus_k C^k_{rsym}(A, M)$, where $C^{k+1}_{rsym}(A, M) = \mathrm{Hom}(A \otimes \wedge^k(A), M), k \geq 0$, with a pre-simplicial structure. The respective cohomologies can be "almost" obtained by the derived functor formalism. The exact meaning of the word "almost" can be found in Sec. 2.5. Roughly speaking, this means that one should be more careful while considering the small degree cohomologies. If we take $C^0_{rsym}(A, M)$ as M, then we should consider the operator d_{rsym} with a cubic condition $d^3_{rsym} = 0$ [17]. We prefer taking $C^0_{rsym}(A, M)$ as $Ker\, d^2_{rsym}$ on M, i.e., $C^0_{rsym}(A, M) := M^{l.ass} := \{m \in M : (m, a, b) = 0, \forall a, b \in A\}$. Then with any $m \in M$, we can associate 2-right-symmetric cocycles, $\nabla(m) : (a, b) \mapsto (m, a, b)$. We call such cocycles standard. If

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$m \in M^{l.ass}$, then $\nabla(m) = 0$. If $m \in M$, then the cohomological class $[\nabla(m)] = 0$, because $\nabla(m) = d\omega$, where $\omega(a) = [a, m]$. Moreover, this is true if M is a submodule of some right-symmetric A-module $\tilde{M}$, and $m \in \tilde{M}$ is such that $[a, m] = a \circ m - m \circ a \in M, \forall a \in M$. If $\tilde{m} \in \tilde{M}$ is such that $d_{rsym}\tilde{m}(a) \notin M$, then $\nabla(\tilde{m})$ can give a nontrivial class of 2-right-symmetric cocycles in $H^2_{rsym}(A, M)$. For example, the Osborn 2-right-symmetric cocycles for $A = W_1^{rsym}(m), p > 0$, that appear in constructing simple Novikov algebras,

$$(u\partial, v\partial) \mapsto x^{p^m-1}uv\partial,$$

$$(u\partial, v\partial) \mapsto x^{p^m-2}uv\partial,$$

are $\nabla(x^{p^m+1}\partial)$, and $\nabla(x^{p^m}\partial)$, respectively.

If $k > 0$, the right-symmetric cohomology $H^{k+1}_{rsym}(A, M)$ is isomorphic to the Chevalley–Eilenberg cohomology $H^k_{lie}(A, C^1(A, M))$, where the A^{lie}-module structure on $C^1(A, M)$ is given in a special way: $[a, f](b) = -d_{rsym}f(b, a)$. We also endow the right-symmetric universal enveloping algebra with a Hopf algebraic structure. This allows us to consider cup products, which are very useful in cocycle constructions.

The second cohomology space $H^2_{rsym}(A, A)$ is interpreted as the space of right-symmetric deformations. We calculate the right-symmetric cohomology of the matrix algebra gl_n, $p = 0$. We prove that, in the category of irreducible antisymmetric gl_n^{rsym}-modules, the nontrivial cohomology appears only in the case of $M = (gl_n)_{anti}$. Moreover, the right-symmetric cohomology of gl_n^{rsym} in $(gl_n)_{anti}$ can be reduced to the Chevalley–Eilenberg cohomology of the Lie algebra gl_n with coefficients in the trivial module:

$$H^{k+1}_{rsym}(gl_n, (gl_n)_{anti}) \cong H^k_{lie}(gl_n, \mathcal{K}), \ k > 0.$$

In particular, $H^{k+1}_{rsym}(gl_n, \mathcal{K}) = 0$, $k \geq 0$. We also calculate the right-symmetric cohomology of gl_n with coefficients in the regular module. These results show that gl_n has $(n^2 - 1)$-parametrical nontrivial right-symmetric deformations. Any formal right-symmetric deformation of gl_n is equivalent to the deformations given by

$$(a, b) \mapsto a \circ b + t\, tr\, b\,[X, a], \ \ X \in sl_2.$$

One can choose an extension in yet another way:

$$(a, b) \mapsto a \circ b + t X \circ ((tr\, a)b - tr\,(a \circ b) + (tr\, b)a)$$

$$+t^2\{tr\, a\, tr\, b - (tr\, a \circ b)^2 X^2 - (tr\, a\, tr(X \circ b))X - (tr\,(a \circ X) tr\, b)X\} + \cdots.$$

We prove that the right-symmetric cohomology of $A = W_n^{rsym}$ with coefficients in antisymmetric modules can also be reduced to the Chevalley–Eilenberg cohomology of the Lie algebra W_n. As it turns out, the space $H^2_{rsym}(A, A)$ for $A = W_n$, $p = 0$, or $A = W_n(\mathrm{m})$, $p > 0$, is too large, and mainly this happens because the space of right-symmetric derivations is large. There is an imbedding

$$Z^1_{rsym}(A, A) \otimes H^1_{lie}(A, U) \to H^2_{rsym}(A, A).$$

We prove that $Z^1_{rsym}(A, A)$ has a basis consisting of right-symmetric derivations of two types: ∂_i, $i = 1, \ldots, n$; if $p > 0$, one should also consider derivations $\partial_i^{p^{k_i}}$, $0 \geq k_i < m_i$; and $x_i\partial_j$, $i, j = 1, \ldots, n$. So, any right-symmetric derivation of A has the form $\sum_{i=1}^n u_i\partial_i + \delta(p > 0)\sum_{i=1}^n \sum_{k_i=0}^{m_i-1} \lambda_{i,k_i}\partial_i^{p^{k_i}}$, such that $\partial_i\partial_j(u_s) = 0$, $i, j, s = 1, \ldots, n$, $\lambda_{i,k_i} \in \mathcal{K}$. We formulate a result about the local deformations of W_n, $p = 0$, or $W_n(\mathrm{m})$, $p > 3$. The space $H^2_{rsym}(W_n, W_n)$, $p = 0$, is generated by classes of cocycles of four types. In the case $p > 3$, Steenrod squares also appear. We prove that W_1^{rsym} has exactly one right-symmetric central extension. It can be given by the cocycle

$$(e_i, e_j) \mapsto (j+1)j\delta_{i+j,-1}, \ \ p = 0,$$

$$(e_i, e_j) \mapsto (-1)^i\delta_{i+j,p^m-1}, \ \ p > 0.$$

For $n > 1$, $H^2_{rsym}(W_n, \mathcal{K}) = 0$.

Regarding right-symmetric algebras, Novikov algebras, and some cohomology calculations, see also [13], [18], [22], [23], [3], [4], and [25].

2. Right-Symmetric Algebras and (Co)modules

2.1. Right-symmetric algebras

An algebra A over a field $\mathcal{K}$ with multiplication $(a,b) \mapsto a \circ b$ is called *Lie-admissible* if the vector space A under the commutator $[a,b] = a \circ b - b \circ a$ is a Lie algebra. An algebra A is *right-symmetric* if it satisfies the following identity:

$$(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b), \quad \forall a,b,c \in A.$$

Let $(a,b,c) = a \circ (b \circ c) - (a \circ b) \circ c$ be the asssociator of elements $a,b,c \in A$. In terms of associators the right-symmetric identity is

$$(a,b,c) = (a,c,b), \ \forall a,b,c \in A.$$

The right-symmetric algebra A is Lie-admissible. Similarly, one can define the left-symmetric algebra by the identity

$$(a,b,c) = (b,a,c), \ \forall a,b,c \in A.$$

Categories of left-symmetric algebras and right-symmetric algebras are equivalent. Any left(right)-symmetric algebra will be right(left)-symmetric under a new multiplication, $(a,b) \mapsto b \circ a$.

An element e of a right-symmetric algebra is called a *left unit* if $e \circ a = a$ for any $a \in A$. Denote by $Q_l(A)$ the space of left units. Let $Z_l(A) = \{z \in A : z \circ a = 0, \forall a \in A\}$ be the *left center* of A. Call the space $N_l(A) = Z_l(A) \oplus Q_l(A)$ the semi-center of A. Then $[N_l(A), N_l(A)] \subseteq Z_l(A)$. An algebra A is called *(left)* unital if it has nontrivial left units.

Any associative algebra is a right-symmetric algebra. In such cases, we will use the notation A^{ass} if we consider A as an associative algebra and A^{rsym} if we consider A as a right-symmetric algebra. Similarly, for a right-symmetric algebra A, the notation A^{rsym} will mean that we use only the right-symmetric structure on A, and A^{lie} stands for a Lie-algebra structure under the commutator $(a,b) \mapsto [a,b]$.

The matrix algebra gl_n gives us an example of a unital right-symmetric algebra.

Less trivial examples appear in the consideration of Witt algebras. The algebra $W_n, p = 0$, and $W_n(\mathrm{m})$ defined below has not only right-symmetric multiplication $(a,b) \mapsto a \circ b$, but also one more multiplication, $(a,b) \mapsto a * b$, which satisfies the following identities:

$$a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b = 0,$$

$$a * (b * c) - b * (a * c) = 0,$$

$$a \circ (b * c) - b * (a \circ c) = 0,$$

$$(a * b - b * a - a \circ b - b \circ a) * c = 0,$$

$$(a \circ b - b \circ a) * c + a * (c \circ b) - (a * c) \circ b - b * (c \circ a) + (b * c) \circ a = 0.$$

Let

$$U = k[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]] = \left\{ x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i} : \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbf{Z}, i = 1, \ldots, n \right\}$$

be the algebra of Laurent power series if the main field k has characteristic 0, and

$$U = O_n(\mathrm{m}) = \left\{ x^{(\alpha)} = \prod_i x_i^{(\alpha_i)} : \alpha = (\alpha_1, \ldots, \alpha_n), 0 \le \alpha_i < p^{m_i}, i = 1, \ldots, n \right\}$$

be the divided power algebra if char$*/ \, k = p > 0$. Recall that $O_n(\mathrm{m})$ is p^m-dimensional and the multiplication is given by

$$x^{(\alpha)} x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{(\alpha+\beta)},$$

where $m = \sum_i m_i$ and

$$\binom{\alpha+\beta}{\alpha} = \prod_i \binom{\alpha_i+\beta_i}{\alpha_i}, \quad \binom{n}{l} = \frac{n!}{l!(n-l)!}, \; n, l \in \mathbf{Z}_+.$$

Let $\varepsilon_i = (0, \ldots, \underset{i}{1}, \ldots, 0)$. Define ∂_i as a derivation of U with

$$\partial_i(x^\alpha) = \alpha_i x^{\alpha-\varepsilon_i}, \; p = 0,$$

$$\partial_i(x^{(\alpha)}) = x^{(\alpha-\varepsilon_i)}, \; p > 0.$$

Endow the space of derivations $Der\,U = \{\sum_i u_i\partial_i : u_i \in U\}$ with multiplications:

$$u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i,$$

$$u\partial_i * v\partial_j = \partial_i(u)v\partial_j.$$

We denote the algebra obtained as $W_n^{rsym}(\mathbf{m})$. If $p = 0$, this notation will be reduced to W_n^{rsym}. If $n \not\equiv 0(\bmod p)$, then an element $e = \sum_i x_1\partial_i/n$ is a left unit of $W_n^{rsym}(m)$. If $n = 1$, then $a \circ b = a * b$. Thus the algebra $A = W_1^{rsym}(\mathbf{m})$, in addition to the right-symmetry condition, satisfies the following identity:

$$a \circ (b \circ c) = b \circ (a \circ c), \; \forall a, b, c \in A.$$

Such algebras are called *Novikov algebras* [1]. Note that the Novikov algebra $W_1(m)$ is unital. If a right-symmetric algebra A is a Novikov algebra, we will use the notation A^{nov}.

2.2. Right-symmetric modules and comodules

A vector space M is said to be a *module* over a right-symmetric algebra A if it is endowed with a right action

$$M \times A \to M, \quad (m, a) \mapsto m \circ a$$

and a left action

$$A \times M \to M, \quad (a, m) \mapsto a \circ m,$$

such that

$$m \circ [a, b] - (m \circ a) \circ b + (m \circ b) \circ a = 0,$$

$$(a \circ m) \circ b - a \circ (m \circ b) - (a \circ b) \circ m + a \circ (b \circ m) = 0,$$

for any $a, b \in A$, $m \in M$. We will say that M is an *antisymmetric* A-module if the left action of A is trivial, i.e., $a \circ m = 0$, for any $a \in A$, $m \in M$. For a module M over a right-symmetric algebra A, denote by M_{anti} its antisymmetric A-module: $M_{anti} = M$, $(m, a) \mapsto m \circ a$, $(a, m) \mapsto 0$, for all $m \in M_{anti}, a \in A$.

A right-symmetric A-module M is said to be *special* if the right action satisfies the following condition:

$$m \circ (a \circ b) - (m \circ a) \circ b = 0, \; \forall a, b \in A, \forall m \in M.$$

A special module is *antisymmetric* if $a \circ m = 0$, for all $a \in A$.

Example. For a right-symmetric algebra A, its vector space A can be endowed with the natural structure of an A-module, $(a, m) \mapsto a \circ m$, $(m, a) \mapsto m \circ a$, $a, m \in A$. In such cases we say that $M = A$ is a *regular* A-module. If A is an associative algebra, then the regular module is special.

The functor

$$A^{lie}\text{-module} \to \text{right } A^{lie}\text{-module} \to \text{Antisymmetric } A\text{-module}$$

gives us an equivalence of the category of antisymmetric A-modules to the category of (right) A^{lie}-modules. An antisymmetric A-module corresponding to a right A^{lie}-module M will be denoted by M_{anti}.

Assume that A is an associative algebra A with multiplication $(a,b) \mapsto a \cdot b$. In the last case of A^{rsym}, right-symmetric multiplications can be defined in two ways: by $(a,b) \mapsto a \cdot b$ or by $(a,b) \mapsto b \cdot a$. To be specific, we endow A^{rsym} with the multiplication $(a,b) \mapsto a \cdot b$. For an associative algebra A the functor

$$\text{Right } A^{ass}\text{-module} \to \text{ Antisymmetric special } A^{rsym}\text{-module}$$

gives us an equivalence of the categories of antisymmetric special A^{rsym}-modules and right A^{ass}-modules.

A right-symmetric A -module M can be endowed with the structure of a module over a Lie algebra A^{lie} by the action $[a,m] = a \circ m - m \circ a$. The module thus obtained is denoted by M^{lie}.

Thus, to define a module structure on a vector space M over a right-symmetric algebra A, one should define on M a right module structure over the Lie algebra A^{lie} and endow it with a left action that satisfies (AAM). As mentioned before, the latter can be done trivially by setting $a \circ m = 0$, $\forall a \in A$, $\forall m \in M$.

For a module M over a right-symmetric algebra A, the subspace

$$M^{l.ass} = \{m :\in M : (m,a,b) = 0, \forall a,b \in A\}$$

is called the *left associative invariant* subspace of M, and

$$M^{l.inv} = \{m \in M : m \circ a = 0, \forall a \in A\}$$

is called the *left invariant* subspace of M. If $M = A$ is a *regular module*, then $A^{l.ass}$ is called the *left associative center*. Note that $A^{l.inv}$ coincides with the left center of A. Note that $M^{l.inv}$ is closed under the right action of A. One has

$$M^{l.inv} \subseteq M^{l.ass}.$$

A module M over a (left) unital right-symmetric algebra is called *(left) unital* if

$$e \circ m = m, \quad \forall e \in \dot{Q}_l(A), \ \forall m \in M.$$

and *(left) central* if

$$z \circ m = 0, \quad \forall z \in Z_l(A), \ \forall m \in M.$$

A regular module over a unital right-symmetric algebra is unital and central.

A vector space M is called a *comodule* over a right-symmetric algebra A if there exist a right action

$$M \times A \to M, \quad (m,a) \mapsto m \circ a,$$

and a left action

$$A \times M \to M, \quad (a,m) \mapsto a \circ m,$$

such that

$$[a,b] \circ m - a \circ (b \circ m) + b \circ (a \circ m) = 0,$$

$$-b \circ (m \circ a) + (b \circ m) \circ a - m \circ (a \circ b) + (m \circ a) \circ b = 0,$$

for any $a,b \in A$, $m \in M$. A comodule M is *special* if it satisfies the identity

$$(a \circ b) \circ m = a \circ (b \circ m), \quad \forall a,b \in A, \forall m \in M.$$

A (special) comodule M is called *antisymmetric* if $m \circ a = 0$ for any $a \in A$.

Example. Let A be a right-symmetric algebra, M an A-module, and $M' = \{f : M \to \mathcal{K}\}$ the space of linear functions on M. Set

$$(a \circ f)(m) = f(m \circ a), \ (f \circ a)(m) = f(a \circ m).$$

Then M' under the actions $(a,f)\mapsto a\circ f,\ (f,a)\mapsto f\circ a$ becomes an A-comodule. We check it:

$$\{[a,b]f - a\circ(b\circ f) + b\circ(a\circ f)\}(m)$$
$$= f(m\circ[a,b] - (m\circ a)\circ b + (m\circ b)\circ a) = 0,$$

$$\{-b\circ(f\circ a) + (b\circ f)\circ a - f\circ(a\circ b) + (f\circ a)\circ b\}(m)$$
$$= f(-a\circ(m\circ b) + (a\circ m)\circ b - (a\circ b)\circ m + a\circ(b\circ m)) = 0.$$

The A-comodule A' for the regular module A is called the *coregular* comodule of A. If A is associative, then A' is a special comodule.

For an A-comodule M let

$$M^{r.ass} = \{m\in M : (a,b,m) = 0,\ \forall a,b\in A\}$$

be the *right associative invariant* subspace of M and

$$M^{r.inv} = \{m\in M : a\circ m = 0,\ \forall a\in A\}$$

the *right invariant* subspace of M. Note that $M^{r.inv}$ is closed under the left action of A. One has

$$M^{r.inv}\subseteq M^{r.ass}$$

2.3. Antisymmetric module $C^1_{right}(A,M)$

Proposition 2.1. *The space of linear maps $C^1_{rsym}(A,M) := C^1(A,M) = \{f : A\to M\}$ can be endowed with the structure of an antisymmetric A-module, where the right action is given by*

$$(f\circ a)(b) = f(b)\circ a - f(a\circ b) + b\circ f(a),\ \ a,b\in A.$$

Proof. For $f\in C^1(A,M)$, $a,b,c\in A$, we have

$$(f\circ[b,c])(a) - ((f\circ b)\circ c)(a) + ((f\circ c)\circ b)(a)$$
$$= d_{rsym}f(a,[b,c]) - d_{rsym}([f,b])(a,c) + d_{rsym}([f,c])(a,b)$$
$$= a\circ f([b,c]) - f(a\circ[b,c]) + f(a)\circ[b,c]$$
$$-a\circ[f,b](c) + [f,b](a\circ c) - [f,b](a)\circ c$$
$$+a\circ[f,c](b) - [f,c](a\circ b) + [f,c](a)\circ b$$
$$= a\circ f([b,c]) - f(a\circ[b,c]) + f(a)\circ[b,c]$$
$$-a\circ d_{rsym}f(c,b) + d_{rsym}f(a\circ c,b) - d_{rsym}f(a,b)\circ c$$
$$+a\circ d_{rsym}f(b,c) - d_{rsym}f(a\circ b,c) + d_{rsym}f(a,c)\circ b$$
$$= a\circ \underline{f([b,c])} - f(\underline{\underline{a\circ[b,c]}}) + f\underline{\underline{(a)\circ[b,c]}}$$
$$-a\circ(c\circ f(b)) + a\circ f(c\circ b) - a\circ(f(c)\circ b)$$
$$+(a\circ c)\circ f(b) - f((a\circ c)\circ b) + f(a\circ c)\circ b$$
$$-a\circ f(b)\circ c + f(a\circ b)\circ c - (f(a)\circ b)\circ c$$
$$+a\circ(b\circ f(c)) - a\circ f(b\circ c) + a\circ(f(b)\circ c)$$
$$-(a\circ b)\circ f(c) + f((a\circ b)\circ c) - f(a\circ b)\circ c$$
$$+(a\circ f(c))\circ b - f(a\circ c)\circ b + (f(a)\circ c)\circ b = 0.$$

Thus, $C^1(A,M)$ is a right A^{lie}-module. □

2.4. Universal enveloping algebras of right-symmetric algebras

Consider two copies of A (denote them by A^r, A^l) and the tensor algebra $T(A^r \oplus A^l)$. The algebras A^r, A^l are assumed to be free as k-modules and the tensor algebra $T(A^r \oplus A^l)$ is associative and unital. The elements of A^r and A^l corresponding to $a \in A$ are denoted by r_a and l_a, respectively. Let $U(A)$ be the factor-algebra of $T(A^r \oplus A^l)$ over an ideal J generated by $r_{[a,b]} - r_a r_b + r_b r_a$, $[r_b, l_a] - l_b l_a + l_{a\circ b}$. This algebra can be considered as a *universal enveloping algebra* of a right-symmetric algebra A. Denote by $\breve{U}(A)$ the factor-algebra of $T(A^r \oplus A^l)$ over an ideal $\breve{J}$ generated by $\{r_{a\circ b} - r_a r_b,\ [r_b, l_a] - l_b l_a + l_{a\circ b}\}$. This algebra is called a *special universal enveloping algebra of* A. Note that $J \subset \breve{J}$, since

$$r_{[a,b]} - [r_a, r_b] = \{r_{a\circ b} - r_a r_b\} \ - \ \{r_{b\circ a} + r_b r_a\} \in \breve{J}.$$

Thus, the following is an exact sequences of algebras:

$$0 \to J \to T(A^r \oplus A^l) \to U(A) \to 0,$$

$$0 \to \breve{J} \to T(A^r \oplus A^l) \to \breve{U}(A) \to 0,$$

and

$$0 \to \breve{J}/J \to U(A) \to \breve{U}(A) \to 0.$$

In particular, we can consider $\breve{U}(A)$ as a right $U(A)$-module:

$$\breve{u}\bar{v} = \breve{u}\breve{v},$$

where $\breve{u}$ and $\bar{u}$ are elements of $\breve{U}(A)$ and $U(A)$ corresponding to $u \in T(A^r \oplus A^l)$.

Theorem 2.2. *Let A be a right-symmetric algebra.*

(i) *There exists an equivalence of the categories of A-modules and right $U(A)$-modules. The same is true for A-comodules and left $U(A)$-modules.*

(ii) *The category of special A-modules is equivalent to the category of right $\breve{U}(A)$-modules. The same is true for special A-comodules and left $\breve{U}(A)$-modules.*

Proof. (i) Let $(r, l) : A \to \operatorname{End} M$ be a representation of a right-symmetric algebra A corresponding to an A -module M, i.e.,

$$r : A \to \operatorname{End} M,\ \ a \mapsto r_a,\ \ m r_a = m \circ a,$$

$$l : A \to \operatorname{End} M,\ \ a \mapsto l_a,\ \ m l_a = a \circ m,$$

are the linear operators, such that for any $a, b \in A$,

$$r_{[a,b]} - r_a r_b + r_b r_a = 0, \tag{MAA}$$

$$[r_b, l_a] - l_b l_a + l_{a\circ b} = 0. \tag{AAM}$$

So, any A-module is a right $U(A)$-module and, conversely, any right $U(A)$-module can be considered as an A-module.

A corepresentation $(r^{co}, l^{co}) : A \to \operatorname{End} M$, corresponding to an A-comodule M,

$$r^{co} : A \to \operatorname{End} M,\ a \mapsto r_a,\ r^{co}_a m = a \circ m,$$

$$l^{co} : A \to \operatorname{End} M,\ a \mapsto l_a,\ l^{co}_a m = m \circ a,$$

satisfies the conditions (MAA), (AAM) for r^{co}_a, l^{co}_a. So, any A-comodule is a left $U(A)$-module. Any left $U(A)$-module can be considered as an A-comodule.

(ii) Let M be a special A-module. Then setting

$$mr_a = m \circ a, ml_a = m \circ a,$$

we obtain a right $\breve{U}(A)$-module:

$$m \circ (a \circ b) - (m \circ a) \circ b = 0 \to r_{a \circ b} = r_a r_b.$$

Conversely, with each right $\breve{U}(A)$-module N, one can associate a special A-module N, if one sets $n \circ a := nr_a, a \circ n = nl_a$.

For a special A-comodule M, we note that

$$(a \circ b) \circ m - a \circ (b \circ m) = 0 \Rightarrow r^{co}_{a \circ b} = r^{co}_a r^{co}_b,$$

if $r^{co}_a m = a \circ m$, $l^{co}_a m = m \circ a$. Thus, any special A-comodule is a left $U^{spec}(A)$-module. The converse statement is also obvious. □

2.5. Right-symmetric cohomology as a derived functor

Recall that the factor-images of the element $u \in T(A^r \oplus A^l)$ in $U(A)$ and $\breve{U}(A)$ were denoted by $\bar{u}$ and $\breve{u}$. Consider $\breve{A} = A \oplus < 1 >$ as a right $U(A)$-module:

$$1 \circ \bar{r}_a = r_a \circ 1 = a,\ 1 \circ \bar{l}_a = l_a \circ 1 = a,\ a \circ \bar{r}_b = a \circ b,\ a \circ \bar{l}_b = b \circ a,$$

for all $a \in A$. We endow $\breve{U}(A)$ with the structure of a $U(A)$-module, as in Sec. 2.4. We consider $A \otimes \wedge^k(A) \otimes U(A), k \geq 0$, as a right $U(A)$-module. Then $A \otimes \wedge^k A \otimes U(A)$ is a free $U(A)$-module. Denote its generators by $< a_0, a_1, \dots, a_k >$, where $a_0 \in A, a_1 \wedge \dots \wedge a_k \in \wedge^k A$. We construct the homomorphisms

$$\partial : A \otimes \wedge^k A \otimes U(A) \to A \otimes \wedge^{k-1} A \otimes U(A), k > 0,$$

$$\partial : A \otimes U(A) \to \breve{U}(A),$$

$$\epsilon : \breve{U}(a) \to \breve{A}$$

as follows:

$$\partial \langle a_0, a_1, \dots, a_k \rangle$$

$$= \sum_{i=1}^{k} \{(-1)^{i+1} \langle a_i, a_1, \dots, \hat{a}_i, \dots, a_k \rangle \bar{l}_{a_0}$$

$$+ (-1)^i \langle a_0 \circ a_i, a_1, \dots, \hat{a}_i, \dots, a_k \rangle$$

$$+ (-1)^{i+1} \rangle a_0, a_1, \dots, \hat{a}_i, \dots, a_k \rangle \bar{r}_{a_i} \}$$

$$+ \sum_{i<j} (-1)^{i+1} \langle a_0, a_1, \dots, \hat{a}_i, \dots, a_{j-1}, [a_i, a_j], \dots, a_k \rangle,$$

$$\partial (a_0) = \langle \breve{1} \rangle \bar{l}_{a_0} - \langle \breve{1} \rangle \bar{r}_{a_0},$$

$$\epsilon \breve{1} = \breve{1},\ \epsilon \breve{r}_a = a,\ \epsilon \breve{l}_a = -a.$$

Then the sequence

$$\dots \xrightarrow{\partial} A \otimes \wedge^2 A \otimes U(A) \xrightarrow{\partial} A \otimes A \otimes U(A) \xrightarrow{\partial} A \otimes U(A) \xrightarrow{\partial} \breve{U}(A) \xrightarrow{\epsilon} \breve{A} \to 0$$

is an almost free resolution of the right $U(A)$-module $\breve{A}$. Here the words "almost free" mean that all terms of the resolution except $\breve{U}(A)$ are free right $U(A)$-modules.

Note that

$$\operatorname*{Hom}_{U(A)}(A \otimes \wedge^k A \otimes U(A), M) \cong A \otimes \wedge^k A, \quad k \geq 0,$$

and $\mathrm{Hom}_{U(A)}(\check{U}(A), M)$ consists of $g : \check{U}(A) \to M$ such that

$$g(\check{1})(\bar{r}_a \bar{r}_b - \bar{r}_{a \circ b}) = g(\check{1}(\bar{r}_a \bar{r}_b - \bar{r}_{a \circ b}))$$
$$= g(r_a \check{r}_b - r_{a \circ b}^{\vee}) = g(\check{r}_a \check{r}_b - r_{a \circ b}^{\vee}) = 0.$$

for any $a, b \in A$. So,

$$\operatorname*{Hom}_{U(A)}(\check{U}(A), M) \cong \{m \in M : (m, a, b) = 0\}.$$

Therefore, we can take

$$C^*_{rsym}(A, M) = \oplus_k C^k_{rsym}(A, M),$$
$$C^0_{rsym}(A, M) = \{m \in M : (m, a, b) = 0, \ \forall a, b \in A\},$$
$$C^{k+1}_{rsym}(A, M) = A \otimes \wedge^k A, \quad k \geq 0,$$

as a right-symmetric cochain complex.

These statements will follow from our results on the right-symmetric cohomology in next sections. Our approach is slightly different from Koszul's approach. We will argue in cohomological terms and prove that the right-symmetric cochain complex has a pre-simplicial structure.

Let us mention these results relating to homologies. Let M be a comodule over a right-symmetric algebra A. Endow $M \otimes A$ with the structure of an antisymmetric A-comodule with a left action:

$$b \circ (m \otimes a) = m \circ a \otimes b - m \otimes a \circ b + b \circ m \otimes a.$$

Set

$$C_0^{rsym}(A, M) := M^{r.ass} := \{m \in M : (a, b, m) = 0, \ \forall a, b \in A\},$$
$$C_{k+1}^{rsym}(A, M) = M \otimes A \otimes \wedge^k(A), \quad k \geq 0.$$
$$C_*^{rsym}(A, M) = \oplus_k C_k^{rsym}(A, M).$$

Then $C_*^{rsym}(A, M)$ is a chain complex under the boundary operator

$$\partial : C_{k+1}^{rsym}(A, M) \to C_k^{rsym}(A, M),$$

$$\partial(m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_k)$$
$$= \sum_{i=1}^{k} (-1)^{i+1} \{ m \circ a_0 \otimes a_i \otimes a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_k$$
$$- m \otimes a_0 \circ a_i \otimes a_1 \wedge \hat{a}_i \cdots \wedge a_k$$
$$+ a_i \circ m \otimes a_0 \otimes a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_k \}$$
$$+ \sum_{i<j} (-1)^{i+1} m \otimes a_0 \otimes a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_{j-1} \wedge [a_i, a_j] \wedge \cdots a_k.$$

Moreover, $C_*^{rsym}(A, M)$ has an antisymmetric A-comodule structure with a left action

$$\rho_{co}^{rsym}(x) : C_{k+1}^{rsym}(A, M) \to C_{k+1}^{rsym}(A, M),$$

$$\rho_{co}^{rsym}(x)(m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_k)$$
$$= \sum_{i=1}^{k} (-1)^i (m \circ a_0 \otimes a_i \otimes a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_k$$

$$-m \otimes a_0 \circ a_i \otimes a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_k$$

$$+a_i \circ m \otimes a_0 \otimes a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_k\}$$

$$+\sum_{i<j}(-1)^{i+1} m \otimes a_0 \otimes a_1 \wedge \cdots \wedge a_{i-1} \wedge [x, a_i] \wedge \cdots \wedge a_k,$$

and the isomorphism of A-comodules

$$C_{k+1}^{rsym}(A, M) \cong C_k^{lie}(A, M \otimes A)$$

takes place, which induces an isomorphism of homology spaces

$$H_{k+1}^{rsym}(A, M) \cong H_k^{lie}(A, M \otimes A), \quad k > 0.$$

2.6. Comultiplication of the universal enveloping algebra

Let $U(A)$ be the universal enveloping algebra of a right-symmetric algebra A. As noted in Sec. 2, it can be generated by the elements $r_a, l_a, a \in A$, such that

$$r_{[a,b]} - [r_a, r_b] = 0, \ [l_a, r_b] - l_{a \circ b} + l_b l_a = 0, \ a, b \in A.$$

We define a homomorphism

$$\Delta : U(A) \to U(A) \otimes U(A)$$

by

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(r_a) = r_a \otimes 1 + 1 \otimes (r_a - l_a),$$

$$\Delta(l_a) = l_a \otimes 1.$$

According to the right-symmetry identities,

$$\Delta([r_a, r_b])$$

$$= \Delta(r_a)\Delta(r_b) - \Delta(r_b)\Delta(r_a)$$

$$= r_a r_b \otimes 1 + 1 \otimes (r_a - l_a)(r_b - l_b) - r_b r_a \otimes 1 + 1 \otimes (r_b - l_b)(r_a - l_a)$$

$$= r_{[a,b]} \otimes 1 + 1 \otimes (r_{[a,b]} - l_{[a,b]}$$

$$= \Delta(r_{[a,b]}),$$

$$\Delta([l_a, r_b] - l_{a \circ b} + l_b l_a)$$

$$= (l_a \otimes 1)(r_b \otimes 1 + 1 \otimes (r_b - l_b)) - (r_b \otimes 1 + 1 \otimes (r_b - l_b))(l_a \otimes 1) - l_{a \circ b} \otimes 1 + l_b l_a \otimes 1$$

$$= ([l_a, r_b] - l_{a \circ b} + l_b l_a) \otimes 1 = 0.$$

Thus, the above definition is correct.

Theorem 2.3. *For a right-symmetric algebra A and its universal enveloping algebra $U(A)$, the following diagram is commutative:*

$$\begin{array}{ccc} U(A) & \xrightarrow{\Delta} & U(A) \otimes U(A) \\ \downarrow \Delta & & \downarrow \Delta \otimes 1 \\ U(A) \otimes U(A) & \xrightarrow{1 \otimes \Delta} & U(A) \otimes U(A) \otimes U(A) \end{array}$$

Proof. We must check that

$$(1 \otimes \Delta)\Delta(u) = (\Delta \otimes 1)\Delta(u), \ \forall u \in U(A).$$

We have

$$\begin{aligned}
&(1 \otimes \Delta)\Delta(r_a) \\
&= (r_a \otimes 1 \otimes 1 + 1 \otimes r_a \otimes 1 + 1 \otimes 1 \otimes (r_a - l_a)) - 1 \otimes l_a \otimes 1 \\
&= (r_a \otimes 1 \otimes 1 + 1 \otimes r_a \otimes 1 - 1 \otimes l_a \otimes 1) + 1 \otimes 1 \otimes (r_a - l_a) \\
&= \Delta(r_a) \otimes 1 + 1 \otimes (r_a - l_a) \\
&= (\Delta \otimes 1) \otimes \Delta(r_a),
\end{aligned}$$

$$(1 \otimes \Delta)\Delta(l_a) = l_a \otimes 1 \otimes 1 = (\Delta \otimes 1)\Delta(l_a). \quad \square$$

Similarly, the homomorphism Δ_1 defined below is also a comultiplication:

$$\Delta_1 : U(A) \to U(A) \otimes U(A),$$

$$\Delta_1(1) = 1 \otimes 1,$$

$$\Delta_1(r_a) = (r_a - l_a) \otimes 1 + 1 \otimes r_a,$$

$$\Delta_1(l_a) = 1 \otimes l_a.$$

So, we can construct, for given A-modules M and N, their tensor product $M \otimes N$ with a module structure induced by the comultiplication Δ:

$$(m \otimes n) \circ a = m \circ a \otimes n + m \otimes [n, a],$$

$$a \circ (m \otimes n) = a \circ m \otimes n.$$

Moreover, this is possible for the right-symmetric A-module M and for an A^{lie}-module N. These module structures on the tensor products are associative: if M, N, S are modules over a right symmetric algebra A, then

$$(M \otimes N) \otimes S \cong M \otimes (N \otimes S).$$

Definition. For given modules M, N over a right-symmetric algebra A, a homomorphism of A-modules $M \otimes N \to S$ is called a cup product of M and N.

Denote the image of $m \otimes n$ in S by $m \cup n$. Thus, a bilinear map

$$M \times N \to S, \ (m, n) \mapsto m \cup n,$$

is said to be the cup product (pairing) of M and N to S, if

$$(m \cup n) \circ a = m \circ a \cup n + m \cup [n, a],$$

$$a \circ (m \cup n) = a \circ m \cup n,$$

for any $a \in A, m, n \in M$.

Let

$$C^1(A, M) = \operatorname{Hom}_k(A, M), \ C^k_{lie}(A, M) = \operatorname{Hom}_k(\wedge^k A, M), \ k \geq 0,$$

$$C^{k+1}_{rsym}(A, M) = \operatorname{Hom}_k(A \otimes \wedge^k(A), M), \ k \geq 0.$$

Proposition 2.4. *$C^{k+1}_{rsym}(A,M)$ has an antisymmetric A-module structure, where the right action*

$$(C^{k+1}_{rsym}(A,M)\times A\to C^{k+1}_{rsym}(A,M),\ \ (\psi,x)\mapsto\psi\circ x,$$

is defined by

$$\begin{aligned}&(\psi\circ x)(a_0,a_1,\ldots,a_k)\\&=a_0\circ\psi(x,a_1,\ldots,a_k)-\psi(a_0\circ x,a_1,\ldots,a_k)\\&+\psi(a_0,a_1,\ldots,a_k)\circ x+\sum_{i=1}^{k}\psi(a_0,a_1,\ldots,a_{i-1},[x,a_i],\ldots,a_k),\end{aligned}$$

for $\psi\in C^{k+1}_{rsym}(A,M),\ k\geq 0$.

Proof. Since

$$C^1_{rsym}(A,M)=C^1(A,M),$$

one has an isomorphism of linear spaces

$$G:C^1_{rsym}(A,M)\otimes C^k_{lie}(A,k)\to C^{k+1}_{right}(A,M),\ k\geq 0,$$

$$(G(f\otimes\psi))(a_0,a_1,\ldots,a_k)=f(a_0)\psi(a_1,\ldots,a_k).$$

In Sec. 2.3, we have constructed an antisymmetric right-module structure on $C^1_{right}(A,M)$. The Lie module structure on $C^k_{lie}(A,k)$ over A^{lie} is well known. So, for an antisymmetric A-module structure

$$C^1_{rsym}(A,M)\otimes C^k_{lie}(A,k)=\{f\otimes\phi:f\in C^1_{right}(A,M),\phi\in C^k_{lie}(A,k)\}$$

we have

$$\begin{aligned}&((f\otimes\phi)\circ x)(a_0\otimes(a_1,\ldots,a_k))\\&=((f\circ x)\otimes\psi)(a_0\otimes(a_1,\ldots,a_k))+(f\otimes[\psi,x])(a_0\otimes(a_1,\ldots,a_k))\\&=(f(a_0)\circ x-f(a_0\circ x)+a_0\circ f(x))\otimes\psi(a_1,\ldots,a_k)\\&+f(a_0)\otimes\sum_{i=1}^{k}\psi(a_1,\ldots,[x,a_i],\ldots,a_k).\end{aligned}$$

We see that

$$G\{(f\otimes\psi)\circ x\}=\{G(f\otimes\psi)\}\circ x.$$

Therefore, $(\psi,x)\mapsto\psi\circ x$ gives us a right representation. □

2.7. Right-symmetric modules for W_n^{rsym}

Let

$$\Gamma_n=\{\alpha=(\alpha_1,\ldots,\alpha_n),\alpha_i\in\mathbf{Z},i=1,\ldots,n\}$$
$$\Gamma_n^+=\{\alpha\in\Gamma_n:\alpha_i\geq 0,i=1,\ldots,n\},$$

and

$$\Gamma_n(\mathbf{m})=\{\alpha\in\Gamma_n^+:\alpha_i<p^{m_i},i=1,\ldots,n\},$$

if $p>0,\mathbf{m}=(m_1,\ldots,m_n)$.

Let

$$U=\mathcal{K}[[x_1^{\pm1},\ldots,x_n^{\pm1}]]=\{x^\alpha:\alpha\in\Gamma_n\},$$
$$U^+=\mathcal{K}[[x_1^1,\ldots,x_n^1]]=\{x^\alpha:\alpha\in\Gamma_n^+\},$$

if $p=0$, and

$$U=O_n(\mathbf{m})=\{x^{(\alpha)}:\alpha\in\Gamma_n(\mathbf{m})\},$$

if $p > 0$.

For $p = 0$, set $A = W_n^{rsym}$ if $U = \mathcal{K}[[x^{\pm 1}, \dots, x_n^{\pm 1}]]$, and $A^+ = W_n^{+rsym}$ if $U^+ = \mathcal{K}[[x_1, \dots, x_n]]$. Let A be $W_n^{rsym}(\mathrm{m})$ if $U = O_n(\mathrm{m})$, $p > 0$. The algebras A, A^+ are right-symmetric and U is associative commutative.

Note that U has the structure of an antisymmetric graded A-module. The right action is given by $u \circ a\partial_i = a\partial_i(u)$. The gradings are given by

$$|x^\alpha| = \sum_i \alpha_i, \quad \alpha \in \Gamma_n \ \ (\text{or } \Gamma_n(\mathbf{m}) \ \text{ if } p > 0),$$

$$U = \oplus_k U_k, \ \ U_k = \{u \in U : |u| = k\},$$

$$A = \oplus_k A_k, \ \ A_k = \{a\partial_i : |a| = k+1, i = 1, \dots, n\}, \ A_k \circ A_l \subseteq A_{k+l},$$

$$U \circ U_l \subseteq U_{k+l}, \ \ U_k \circ A_l \subseteq U_{k+l}, k, l \in \mathbf{Z}.$$

Note that $A_0 \cong gl_n^{rsym}$.

Let $\mathcal{A}_0 = \oplus_k A_k$, and $\mathcal{A}^+{}_0 = \oplus_k > 0 A_k^+$ if $p = 0$. Let M be an $\mathcal{A}_0$-module if $p > 0$, and an $\mathcal{A}_0^+$-module if $p = 0$. Define an antisymmetric A-module structure on $U \otimes M_0$ by

$$(u \otimes m) \circ a\partial_i = a\partial_i(u) \otimes m + \sum_{\beta \in \Gamma_n} u\partial^\beta(a) \otimes [m, x^{(\beta)}\partial_i], \ \ p > 0,$$

$$(u \otimes m) \circ a\partial_i = a\partial_i(u) \otimes m + \sum_{\beta \in \Gamma_n^+} (1/\beta!) u\partial^\beta(a) \otimes [m, x^\beta \partial_i], \ p = 0.$$

3. Cohomology of Right-Symmetric Algebras

3.1. Pre-simplicial structures on $C^{*+1}_{rsym}(A, M)$

For a right-symmetric algebra A and its module M, we introduce the structure of a pre-simplicial cochain complex on $C^{*+1}_{rsym}(A, M) = \oplus_{k \geq 0} C^{k+1}_{rsym}(A, M)$, where

$$C^{k+1}_{rsym}(A, M) = \mathrm{Hom}(A \otimes \wedge^k A, M), \ \ k \geq 0.$$

Define linear operators $D_i : C^{*+1}_{rsym}(A, M) \to C^{*+1}_{rsym}(A, M), i = 1, 2, \dots$, by

$$D_i : C^k_{rsym}(A, M) \to C^{k+1}_{rsym}(A, M),$$

$$D_i\psi(a_0, a_1, \dots, a_k)$$

$$= a_0 \circ \psi(a_i, a_1, \dots, \hat{a}_i, \dots, a_k) - \psi(a_0 \circ a_i, a_1, \dots, \hat{a}_i, \dots, a_k)$$

$$+ \psi(a_0, a_1, \dots, \hat{a}_i, \dots, a_k) \circ a_i + \sum_{i<j} \psi(a_0, a_1, \dots, \hat{a}_i, \dots, a_{j-1}, [a_i, a_j], \dots, a_k),$$

$0 \leq k, \ i \leq k,$

$$D_i\psi = 0, \ i > k.$$

Here $\hat{a}$ means that the element a is omitted.

In the next section, we will endow $C^*_{rsym}(A, M) = \oplus_k C^k_{rsym}(A, M)$ with the structure of a cochain complex, where

$$C^k_{rsym}(A, M) = 0, \quad k < 0,$$

$$C^0_{rsym}(A, M) = \{m \in M : (ma)b = m(ab), \quad \forall a, b \in A\}.$$

Theorem 3.1. *The set of endomorphisms $D_i, i = 1, 2, \dots,$ endows $C^{*+1}_{rsym}(A, M) = \oplus_{k>0} C^k_{rsym}(A, M)$ with a pre-simplicial structure:*

$$D_j D_i = D_i D_{j-1},\ i < j,$$

In particular, $d_{rsym} = -\sum_i (-1)^i D_i$ is a coboundary operator on $C^{*+1}_{rsym}(A, M)$:

$${d_{rsym}}^2 = 0.$$

Proof. For $i < j$, $1 < k$, we have

$$D_j D_i \psi(a_0, a_1, \dots, a_k) = X_1 + X_2 + X_3 + X_4,$$

where

$$X_1 = a_0(D_i\psi(a_j, a_1, \dots, \hat{a}_j, \dots, a_k),$$

$$X_2 = -D_i\psi(a_0 \circ a_j, a_1, \dots, \hat{a}_j, \dots, a_k),$$

$$X_3 = D_i\psi(a_0, a_1, \dots, \hat{a}_j, \dots, a_k)a_j,$$

$$X_4 = \sum_{j<s} D_i\psi(a_0, \dots, \hat{a}_j, \dots, a_{s-1}, [a_j, a_s], \dots, a_k).$$

Direct calculations show that

$$X_1 = a_0(a_j\psi(a_i, a_1, \underline{\dots, \hat{a}_i}, \dots, \hat{a}_j, \dots, a_k))$$

$$-a_0\psi(a_j \circ a_i, a_1, \underline{\underline{\dots, \hat{a}_i}}, \dots, \hat{a}_j, \dots, a_k)$$

$$+a_0(\psi(a_j, a_1, \underline{\underline{\underline{\dots, \hat{a}_i}}}, \dots, \hat{a}_j, \dots, a_k)a_i)$$

$$+\sum_{i<s, s\neq j} a_0(\psi(a_j, a_1, \dots, \underset{\sim}{\hat{a}_i, \dots}, [a_i, a_s], \dots, a_k)),$$

$$X_2 = -(a_0 \circ a_j)\psi(a_i, a_1, \underset{\sim}{\dots, \hat{a}_i}, \dots, \hat{a}_j, \dots, a_k)$$

$$+\psi((a_0 \circ a_j) \circ a_i, a_1, \underset{\approx}{\dots, \hat{a}_i}, \dots, \hat{a}_j, \dots, a_k)$$

$$-(\psi(a_0 \circ a_j, a_1, \underset{\underline{\approx}}{\dots, \hat{a}_i}, \dots, \hat{a}_j, \dots, a_k))a_i$$

$$-\sum_{i<s, s\neq j} \psi(a_0 \circ a_j, a_1, \dots, \underset{\approx\approx}{\hat{a}_i, \dots}, [a_i, a_s], \dots, a_k),$$

$$X_3 = +(a_0\psi(a_i, a_1, \dots, \underline{\hat{a}_i, \dots}, \hat{a}_j, \dots, a_k))a_j$$

$$-(\psi(a_0 \circ a_i, a_1, \underset{\smile}{\dots, \hat{a}_i}, \dots, \hat{a}_j, \dots, a_k))a_j$$

$$+((\psi(a_0, a_1, \dots, \underset{\frown}{\hat{a}_i}, \dots, \hat{a}_j, \dots, a_k)a_i)a_j$$

$$+\sum_{i<s, s\neq j} (\psi(a_0, a_1, \dots, \hat{a}_i, \underset{\doteq}{\dots}, a_{s-1}, [a_i, a_s], \dots, a_k))a_j$$

$$+\sum_{j<s} a_0(\psi(a_i, a_1, \dots, \hat{a}_i, \dots, \underset{\equiv\equiv\equiv\equiv}{\hat{a}_j, \dots}, a_{s-1}, [a_j, a_s], \dots, a_k)),$$

$$X_4 = -\sum_{j<s} \psi(a_0 \circ a_i, a_1, \dots, \hat{a}_i, \dots, \underset{\cong\cong\cong\cong}{\hat{a}_j, \dots}, a_{s-1}, [a_j, a_s], \dots, a_k)$$

$$+\sum_{j<s}(\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\underset{\sim\sim\sim\sim}{\ldots},a_{s-1},[a_j,a_s],\ldots,a_k))a_i$$

$$+\sum_{j<s,i<s_1,s<s_1}\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\ldots,\underset{\simeq\simeq\simeq\simeq}{a_{s-1}},[a_j,a_s],\ldots,a_{s_1-1},[a_i,a_{s_1}],\ldots,a_k)$$

$$+\sum_{j<s,i<s_1,s_1<s,s_1\neq j}\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,\ldots,\underset{----}{a_{s_1-1}},[a_j,a_{s_1}],\ldots,a_{s-1},[a_i,a_s],\ldots,a_k)$$

$$+\sum_{j<s,(i<s_1,s_1=s)}\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\underset{====}{\ldots},a_{s-1},[a_i,[a_j,a_s]],\ldots,a_k).$$

Analogously,

$$D_iD_{j-1}\psi(a_0,a_1,\ldots,a_k)=Y_1+Y_2+Y_3+Y_4,$$

where

$$Y_1=a_0(D_{j-1}\psi(a_i,a_1,\ldots,\hat{a}_i,\ldots,a_k),$$

$$Y_2=-D_{j-1}(a_0\circ a_i,a_1,\ldots,\hat{a}_i,\ldots,a_k),$$

$$Y_3=D_{j-1}\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,a_k)a_i,$$

$$Y_4=\sum_{i<s}D_{j-1}\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,a_{s-1},[a_i,a_s],\ldots,a_k).$$

We have

$$Y_1=a_0(a_i\psi(a_j,a_1,\underline{\underline{\underline{\ldots,\hat{a}_i}}},\ldots,\hat{a}_j,\ldots,a_k))$$

$$-a_0\psi(a_i\circ a_j,a_1,\underline{\underline{\ldots}},\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_k)$$

$$+a_0(\psi(a_i,a_1,\underline{\ldots,\hat{a}_i},\ldots,\hat{a}_j,\ldots,a_k)a_j)$$

$$+\sum_{j<s}a_0\psi(a_i,a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\underset{====}{\ldots},a_{s-1},[a_j,a_s],\ldots,a_k),$$

$$Y_2=-(a_0\circ a_i)\psi(a_j,a_1,\underset{\sim\sim\sim\sim}{\ldots},\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_k)$$

$$+\psi((a_0\circ a_i)\circ a_j,\underset{\approx\approx\approx}{a_1,\ldots},\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_k)$$

$$-\psi(a_0\circ a_i,a_1,\underset{\sim\sim\sim}{\ldots,\hat{a}_i},\ldots,\hat{a}_j,\ldots,a_k)a_j$$

$$-\sum_{j<s}\psi(a_0\circ a_i,a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\underset{\cong\cong\cong\cong}{\ldots},a_{s-1},[a_j,a_s],\ldots,a_k),$$

$$Y_3=(a_0\psi(a_j,a_1,\ldots,\underline{\underline{\underline{\hat{a}_i,\ldots}}},\hat{a}_j,\ldots,a_k))a_i$$

$$-(\psi(a_0\circ a_j,a_1,\ldots,\underset{\approx\approx\approx}{\hat{a}_i},\ldots,\hat{a}_j,\ldots,a_k))a_i$$

$$+((\psi(a_0,a_1,\ldots,\underset{\frown\frown\frown}{\hat{a}_i,\ldots},\hat{a}_j,\ldots,a_k)a_j)a_i$$

$$+\sum_{j<s}(\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\underset{\sim\sim\sim\sim}{\ldots},a_{s-1},[a_j,a_s],\ldots,a_k))a_i.$$

We represent Y_4 as a sum:

$$Y_4=Y_{4,1}+Y_{4,2}+Y_{4,3},$$

where

$$Y_{4,1}=\sum_{i<s<j}D_{j-1}\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,a_{s-1},[a_i,a_s],\ldots,a_k),$$

$$Y_{4,2} = D_{j-1}\psi(a_0, a_1, \dots, \hat{a}_i, \dots, a_{j-1}, [a_i, a_j], \dots, a_k),$$

$$Y_{4,3} = \sum_{j<s} D_{j-1}\psi(a_0, a_1, \dots, \hat{a}_i, \dots, a_{s-1}, [a_i, a_s], \dots, a_k).$$

These elements can be expressed in the following ways:

$$\begin{aligned}
Y_{4,1} = & \sum_{i<s<j} a_0(\psi(a_j, a_1, \dots, \hat{a}_i, \dots, \underline{a_{s-1}}, [a_i, a_s], \dots, \hat{a}_j, \dots, a_k)) \\
& - \sum_{i<s<j} \psi(a_0 \circ a_j, a_1, \dots, \hat{a}_i, \dots, \underline{a_{s-1}}, [a_i, a_s], \dots, \hat{a}_j, \dots, a_k) \\
& + \sum_{i<s<j} (\psi(a_0, a_1, \dots, \hat{a}_i, \dots, \underline{a_{s-1}, [a_i, a_s]}, \dots, \hat{a}_j, \dots, a_k))a_j \\
& + \sum_{i<s_1<j<s} \psi(a_0, a_1, \dots, \hat{a}_i, \dots, a_{s_1-1}, \underline{[a_i, a_{s_1}]}, \dots, \hat{a}_j, \dots, a_{s-1}, [a_j, a_s], \dots, a_k),
\end{aligned}$$

$$\begin{aligned}
Y_{4,2} = & +a_0(\psi([a_i, a_j], a_1, \underline{\dots}, \hat{a}_i, \dots, \hat{a}_j, \dots, a_k)) \\
& - \psi(a_0 \circ [a_i, a_j], a_1, \underline{\dots}, \hat{a}_i, \dots, \hat{a}_j, \dots, a_k) \\
& + (\psi(a_0, a_1, \dots, \hat{a}_i, \underline{\dots}, \hat{a}_j, \dots, a_k))[a_i, a_j] \\
& + \sum_{j<s} \psi(a_0, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \underline{\dots}, a_{s-1}, [[a_i, a_j], a_s], \dots, a_k),
\end{aligned}$$

$$\begin{aligned}
Y_{4,3} = & \sum_{j<s} a_0(\psi(a_j, a_1, \dots, \hat{a}_i, \dots, \underline{\hat{a}_j, \dots}, a_{s-1}, [a_i, a_s], \dots, a_k)) \\
& - \sum_{j<s} \psi(a_0 \circ a_j, a_1, \dots, \hat{a}_i, \dots, \underline{\hat{a}_j, \dots}, a_{s-1}, [a_i, a_s], \dots, a_k) \\
& + \sum_{j<s} (\psi(a_0, a_1, \dots, \hat{a}_i, \dots, \underline{\hat{a}_j, \dots}, a_{s-1}, [a_i, a_s], \dots, a_k))a_j \\
& + \sum_{j<s<s_1} \psi(a_0, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{s-1}, \underline{[a_i, a_s]}, \dots, a_{s_1-1}, [a_j, a_{s_1}], \dots, a_k) \\
& + \sum_{j<s} \psi(a_0, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \underline{\dots}, a_{s-1}, [a_j, [a_i, a_s]], \dots, a_k) \\
& + \sum_{j<s_1<s} \psi(a_0, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{s_1-1}, \underline{[a_i, a_{s_1}]}, \dots, a_{s-1}, [a_j, a_s], \dots, a_k)
\end{aligned}$$

Using the right-symmetric identity for the expressions underlined, in a similar way we obtain that

$$D_j D_i = D_i D_{j-1}, \qquad i < j. \qquad \square$$

3.2. Cohomology of right-symmetric algebras and Cartan's formulas

In the previous section, we proved that $C^{*+1}_{rsym}(A, M) = \oplus_{k>0} C^k_{rsym}(A, M)$ is a cochain complex under coboundary operator d_{rsym}, such that

$$\begin{aligned}
& d_{rsym}\psi(a_0, a_1, \dots, a_k) \\
& = -\sum_{i=1}^{k} (-1)^i a_0 \circ (\psi(a_i, a_1, \dots, \hat{a}_i, \dots, a_k))
\end{aligned}$$

$$+\sum_{i=1}^{k}(-1)^i\psi(a_0\circ a_i,a_1,\ldots,\hat{a}_i,\ldots,a_k)$$

$$\sum_{1\le i<j\le k}(-1)^{i+1}\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,[a_i,a_j],\ldots,a_k)$$

$$-\sum_{i=1}^{k}(-1)^i(\psi(a_0,a_1,\ldots,\hat{a}_i,\ldots,a_k))\circ a_i,$$

$$\psi\in C^k(A,M), 0<k.$$

For $m\in M$, define $d_{rsym}\in C^1_{rsym}(A,M)$:

$$d_{rsym}m(a)=a\circ m-m\circ a.$$

It is easy to see that

$$d^2_{rsym}m(a,b)$$

$$=a\circ d_{rsym}m(b)-d_{rsym}m(a\circ b)+d_{rsym}m(a)\circ b$$

$$=a\circ\underline{(b\circ m)}-a\circ\underline{\underline{(m\circ b)}}-\underline{(a\circ b)}\circ m+m\circ(a\circ b)+\underline{\underline{(a\circ m)}}\circ b-(m\circ a)\circ b$$

$$=(a,b,m)-(a,m,b)+m\circ(a\circ b)-(m\circ a)\circ b.$$

Thus, according to the right-symmetric identity,

$$d^2_{rsym}m(a,b)=m\circ(a\circ b)-(m\circ a)\circ b. \tag{1}$$

Two conclusions follow from this. First, taking the *subspace of left associative invariants*

$$M^{l.ass}=\{m\in M:(m,a,b)=0,\ \forall a,b\in A\}$$

as a 0-cochain subspace $C^0_{rsym}(A,M)$, we obtain a *cochain complex*

$$C^*_{rsym}(A,M)=\oplus_{k\ge 0}C^k_{rsym}(A,M)$$

under the coboundary operator d_{rsym}. The second conclusion will be discussed in the next section in the construction of standard 2-cocycles.

Let

$$Z^*_{rsym}(A,M)=\oplus_k Z^k_{rsym}(A,M),$$

$$Z^k_{rsym}(A,M)=\{\psi\in C^k_{rsym}(A,M):d_{rsym}\psi=0\}$$

be the spaces of right-symmetric cocycles,

$$B^*_{rsym}(A,M)=\oplus_k B^k_{rsym}(A,M),$$

$$B^k_{rsym}(A,M)=\{d_{rsym}\omega:\omega\in C^{k-1}_{rsym}(A,M)\}$$

be the spaces of right-symmetric coboundaries, and

$$H^*_{rsym}(A,M)=\oplus_k H^k_{rsym}(A,M),$$

$$H^k_{rsym}(A,M)=Z^k_{rsym}(A,M)/B^k_{rsym}(A,M)$$

be the right-symmetric cohomology spaces.

Definitions. For any $x \in A$, the interior product endomorphism $i(x)$ of $C^*_{rsym}(A, M)$ is defined by

$$i(x) : C^{k+1}_{rsym}(A, M) \to C^k_{rsym}(A, M),$$

$$i(x)\psi(a_0, \dots, a_{k-1}) = \psi(a_0, x, a_1, \dots, a_{k-1}), \quad k > 0,$$

$$i(x)\psi = 0, \quad \psi \in C^1_{rsym}(A, M).$$

Let $\rho_{lie} : A^{lie} \to C^{*+1}_{rsym}(A, M)$ be a representation of the Lie algebra A^{lie} corresponding to the antisymmetric representation

$$\rho_{rsym} : A \to C^{*+1}_{rsym}(A, M), \ \rho_{rsym}(x)\psi = 0, \ \psi\rho_{rsym}(x) = \psi \circ x$$

constructed in Proposition 2.4:

$$(\rho_{lie}(x)\psi)(a_0, a_1, \dots, a_k)$$
$$= -a_0 \circ \psi(x, a_1, \dots, a_k) + \psi(a_0 \circ x, a_1, \dots, a_k)$$
$$-\psi(a_0, a_1, \dots, a_k) \circ x - \sum_{i=1}^{k} \psi(a_0, a_1, \dots, a_{i-1}, [x, a_i], \dots, a_k).$$

Recall that

$$\rho_{lie}(x)\psi = -\psi\rho_{lie}(x) = -\psi \circ x = -\psi\rho_{rsym}(x).$$

Proposition 3.2 (Cartan's formulas). *Consider $C^{*+1}_{rsym}(A, M) := \oplus_k C^{k+1}_{rsym}(A, M)$ as an A^{lie}-module. For the linear operators on $C^{*+1}_{rsym}(A, M)$, the following relations hold:*

(i) $$i(x)D_l = D_{l-1}i(x), \ l > 1,$$

(ii) $$i(x)D_1 = -\rho_{lie}(x),$$

(iii) $$\rho_{lie}[x, y] = [\rho_{lie}(x), \rho_{lie}(y)],$$

(iv) $$[i(x), \rho_{lie}(y)] = -i([x, y]),$$

(v) $$d_{rsym}i(x) + i(x)d_{rsym} = -\rho_{lie}(x),$$

Proof. (i)

$$i(x)D_l\psi(a_0, \dots, a_{k-1})$$
$$= D_l\psi(a_0, x, a_1, \dots, a_{k-1})$$
$$= a_0 \circ \psi(a_{l-1}, x, a_1, \dots, \hat{a_{l-1}}, \dots, a_{k-1})$$
$$-\psi(a_0 \circ a_{l-1}, x, a_1, \dots, \hat{a_{l-1}}, \dots, a_{k-1})$$
$$+\psi(a_0, x, a_1, \dots, \hat{a_{l-1}}, \dots, a_{k-1}) \circ a_{l-1}$$
$$+ \sum_{l-1<j} \psi(a_0, x, a_1, \dots, \hat{a_{l-1}}, \dots, a_{j-1}, [a_{l-1}, a_j], \dots, a_{k-1})$$
$$= a_0 \circ i(x)\psi(a_{l-1}, a_1, \dots, \hat{a_{l-1}}, \dots, a_{k-1})$$
$$-i(x)\psi(a_0 \circ a_{l-1}, a_1, \dots, \hat{a_{l-1}}, \dots, a_{k-1})$$
$$+i(x)\psi(a_0, a_1, \dots, \hat{a_{l-1}}, \dots, a_{k-1}) \circ a_{l-1}$$
$$+ \sum_{l-1<j} i(x)\psi(a_0, a_1, \dots, \hat{a_{l-1}}, \dots, a_{j-1}, [a_i, a_j], \dots, a_{k-1})$$
$$= D_{l-1}i(x)\psi(a_0, \dots, a_{k-1}).$$

(ii)

$$
\begin{aligned}
&i(x)D_1\psi(a_0,\dots,a_{k-1})\\
&= D_1\psi(a_0,x,a_1,\dots,a_{k-1})\\
&= a_0\circ\psi(x,a_1,\dots,a_{k-1})-\psi(a_0\circ x,a_1,\dots,a_{k-1})\\
&+\psi(a_0,a_1,\dots,a_{k-1})\circ x+\sum_{0<j}\psi(a_0,a_1,\dots,a_{j-1},[x,a_j],\dots,a_{k-1})\\
&= (\psi\rho_{rsym}(x))(a_0,\dots,a_{k-1}).
\end{aligned}
$$

(iii) Proposition 2.4.

(iv) We obtain

$$
\begin{aligned}
&-\{(i(x)\rho_{lie}(y))\psi\}(a_0,a_1,\dots,a_{k-1})\\
&= (\psi\circ y)(a_0,x,a_1,\dots,a_{k-1})\\
&= a_0\circ(\psi(\underline{y,x},a_1,\dots,a_{k-1})-\psi(a_0\circ y,\underline{\underline{x,a_1}},\dots,a_{k-1})\\
&+\psi(a_0,x,a_1,\dots,a_{k-1})\circ y\\
&+\psi(a_0,[x,y],a_1,\dots,a_{k-1})+\sum_{i=1}^{k-1}\psi(a_0,x,a_1,\dots,a_{i-1},[a_i,y],\dots,a_{k-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\{(\rho_{lie}(y)i(x))\psi\}(a_0,a_1,\dots,a_{k-1})\\
&= \{(i(x)\psi)\circ y\}(a_0,a_1,\dots,a_{k-1})\\
&= a_0\circ\{(i(x)\psi)(y,a_1,\dots,a_{k-1})\}-(i(x)\psi)(a_0\circ y,a_1,\dots,a_{k-1})\\
&+\{(i(x)\psi)(a_0,a_1,\dots,a_{k-1})\}\circ y\\
&+\sum_{i=1}^{k-1}(i(x)\psi)(a_0,a_1,\dots,a_{i-1},[a_i,y],\dots,a_{k-1})\\
&= a_0\circ(\psi(\underline{y,x},a_1,\dots,a_{k-1}))-\psi(a_0\circ y,\underline{\underline{x,a_1}},\dots,a_{k-1})\\
&+\psi(a_0,x,a_1,\dots,a_{k-1})\circ y\\
&+\sum_{i=1}^{k-1}\psi(a_0,x,a_1,\dots,a_{i-1},[a_i,y],\dots,a_{k-1}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\{(-i(x)\rho_{lie}(y)+\rho_{lie}(y)i(x))\psi\}(a_0,a_1,\dots,a_{k-1})\\
&= (i[x,y]\psi)(a_0,a_1,\dots,a_{k-1}).
\end{aligned}
$$

(v) According to (i) and (ii),

$$
\begin{aligned}
i(x)d_{rsym} &= i(x)D_1+\sum_{l>1}(-1)^{l+1}i(x)D_l\\
&= -\rho_{lie}(x)+\sum_{l>1}(-1)^{l+1}D_{l-1}i(x)\\
&= -\rho_{lie}(x)-\sum_{l>0}(-1)^{l+1}D_l i(x).
\end{aligned}
$$

Thus,

$$
i(x)d_{rsym}+d_{rsym}i(x)=-\rho_{lie}(x).
$$

3.3. Long exact cohomological sequences

The following theorem follows from standard homological results.

Theorem 3.3. *Let A be a right-symmetric algebra and*

$$0 \to M \to T \to S \to 0$$

a short exact sequence of right-symmetric A-modules. Then the following is an exact sequence of right-symmetric cohomology spaces:

$$0 \to Z^0_{rsym}(A,M) \to Z^0_{rsym}(A,T) \to Z^0_{rsym}(A,S) \xrightarrow{\delta}$$

$$Z^1_{rsym}(A,M) \to Z^1_{rsym}(A,T) \to Z^1_{rsym}(A,S) \xrightarrow{\delta} H^2_{rsym}(A,M) \to \cdots$$

$$\to H^k_{rsym}(A,M) \to H^k_{rsym}(A,T) \to H^k_{rsym}(A,S) \xrightarrow{\delta} H^{k+1}_{rsym}(A,M) \to \cdots$$

Here δ is a connecting homomorphism:

$$\delta[\psi] = [d_{rsym}\phi], \quad [\psi] \in H^k_{rsym}(A,S), k > 1,$$

$$\delta\psi_1 = [d_{rsym}\phi_1], \quad \psi_1 \in Z^i_{rsym}(A,S), i = 0,1,$$

where $\phi \in Z^k_{rsym}(A,T)$ is a representative of the cohomological class $[\psi]$ and $\phi_1 \in Z^i_{rsym}(A,T)$ moves to ψ_1 under the natural homomorphism $Z^i_{rsym}(A,T) \to Z^i_{rsym}(A,S), i = 0,1$.

Define a homomorphism $\nabla : S^{l.ass} \to Z^2_{rsym}(A,M)$ as a composition

$$\nabla : C^0_{rsym}(A,S) \xrightarrow{d_{rsym}} B^1_{rsym}(A,S) \xrightarrow{\delta} Z^2_{rsym}(A,M), \quad s \mapsto d_{rsym}s \mapsto \delta(d_{rsym}s).$$

Then

$$\nabla(\tilde{m})(a,b) = \tilde{m} \circ (a \circ b) - (\tilde{m} \circ a) \circ b.$$

In particular, there exist homomorphisms

$$\delta : S^{l.ass} \to Z^1_{rsym}(A,M), \quad \delta(s) : a \mapsto [a,s],$$

$$\delta : S^{l.inv} \to Z^1_{rsym}(A,M), \quad -\delta(s) : a \mapsto a \circ s.$$

3.4. Connections between the right-symmetric cohomology and the Chevalley–Eilenberg cohomology

Recall that for any A^{lie}-module Q, the standard representation $\varrho : A^{lie} \to C^*_{lie}(A,Q)$ is given by

$$\varrho(x)\psi(a_1,\ldots,a_k)$$

$$= [x, \psi(a_1,\ldots,a_k)] - \sum_{i=1}^{k} \psi(a_1,\ldots,a_{i-1},[x,a_i],\ldots,a_k),$$

where $\psi \in C^k_{lie}(A,Q)$ and $(x,q) \mapsto [x,q]$ is a representation corresponding to the Lie module Q.

Theorem 3.4. *Let A be a right-symmetric algebra and M be an A-module. An operator*

$$F : C^k_{lie}(A, C^1(A,M)) \to C^{k+1}_{rsym}(A,M), \quad k > 0, \tag{2}$$

defined by the rule

$$F\psi(a_0,a_1,\ldots,a_k) = -\psi(a_1,\ldots,a_{k-1})(a_0)$$

induces an isomorphism of A^{lie}-modules. Moreover, F induces an isomorphism of cochain complexes $C^{*+1}_{rsym}(A, M)$ and $C^{*}_{lie}(A, C^1(A, M))$. In particular,

$$H^{k+1}_{rsym}(A, M) \cong H^{k}_{lie}(A, C^1(A, M)), \ k > 0. \tag{3}$$

The following sequence is exact:

$$0 \to Z^0_{rsym}(A, M) \to C^0_{rsym}(A, M) \to H^0_{lie}(A, C^1(A, M)) \to H^1_{rsym}(A, M) \to 0. \tag{4}$$

Proof. Prove that for any $x \in A$, $k > 0$, the following diagram is commutative:

$$\begin{array}{ccc} C^k_{lie}(A, C^1(A, M)) & \xrightarrow{\varrho(x)} & C^k_{lie}(A, C^1(A, M)) \\ \downarrow F & & \downarrow F \\ C^{k+1}_{rsym}(A, C^1(A, M)) & \xrightarrow{\rho_{lie}(x)} & C^{k+1}_{rsym}(A, M) \end{array}$$

For $\psi \in C^k_{lie}(A, C^1(A, M))$ we have

$$F\{\varrho(x)\psi\}(a_0, a_1, \ldots, a_{k+1}) = -\varrho(x)\psi(a_1, \ldots, a_{k+1})(a_0)$$

$$= -\{x \circ (\psi(a_1, \ldots, a_k))\}(a_0) + \sum_{i=1}^{k} \psi(a_1, \ldots, a_{i-1}, [x, a_i], \ldots, a_k)(a_0)$$

$$= +\{(\psi(a_1, \ldots, a_k)) \circ x\}(a_0) + \sum_{i=1}^{k} \psi(a_1, \ldots, a_{i-1}, [x, a_i], \ldots, a_k)(a_0)$$

$$= -a_0 \circ \psi(x, a_1, \ldots, a_k) + \psi(a_0 \circ x, a_1, \ldots, a_k) - \psi(a_0, a_1, \ldots, a_k) \circ x$$

$$- \sum_{i=1}^{k} \psi(a_0, a_1, \ldots, [x, a_i], \ldots, a_k)$$

$$= -(\psi \rho_{rsym}(x))(a_0, a_1, \ldots, a_k)$$

$$= \{(F\psi)\rho_{rsym}(x)\}(a_1, \ldots, a_k)(a_0).$$

Thus, $F : C^k_{lie}(A, C^1(A, M)) \to C^{k+1}_{rsym}(A, M)$ is a homomorphism of A^{lie}-modules. It is obvious that F has no kernel and F is an epimorphism.

Now, prove that for any $k \geq 0$ the following diagram is commutative:

$$\begin{array}{ccc} C^k_{lie}(A, C^1(A, M)) & \xrightarrow{d_{lie}} & C^{k+1}_{lie}(A, C^1(A, M)) \\ \downarrow F & & \downarrow F \\ C^{k+1}_{rsym}(A, C^1(A, M)) & \xrightarrow{d_{rsym}} & C^{k+2}_{rsym}(A, M) \end{array}$$

For $\psi \in C^k_{lie}(A, C^1(A, M))$, we have

$$F(d_{lie}\psi)(a_0, a_1, \ldots, a_{k+1}) = d_{lie}\psi(a_1, \ldots, a_{k+1})(a_0)$$

$$= \sum_{1 \leq i < j \leq k+1} (-1)^i \psi(a_1, \ldots, \hat{a}_i, \ldots, [a_i, a_j], \ldots, a_{k+1})(a_0)$$

$$- \sum_{i=1}^{k+1} (-1)^i [a_i, \psi(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1})](a_0)$$

$$= \sum_{1 \leq i < j \leq k+1} (-1)^{i+1} F\psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, [a_i, a_j], \ldots, a_{k+1})$$

$$+ \sum_{i=1}^{k+1} (-1)^i d_{rsym}(\psi(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}))(a_0, a_i)$$

$$= \sum_{1\le i<j\le k+1} (-1)^{i+1} F\psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, [a_i, a_j], \ldots, a_{k+1})$$

$$+ \sum_{i=1}^{k+1} (-1)^i a_0 \circ (\psi(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}))(a_i)$$

$$- \sum_{i=1}^{k+1} (-1)^i (\psi(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}))(a_0 \circ a_i)$$

$$+ \sum_{i=1}^{k+1} (-1)^i (\psi(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1})(a_0)) \circ a_i$$

$$= \sum_{1\le i<j\le k+1} (-1)^{i+1} F\psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, [a_i, a_j], \ldots, a_{k+1})$$

$$- \sum_{i=1}^{k+1} (-1)^i a_0 \circ (F\psi(a_i, a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}))$$

$$+ \sum_{i=1}^{k+1} (-1)^i F\psi(a_0 \circ a_i, a_1, \ldots, \hat{a}_i, \ldots, a_{k+1})$$

$$- \sum_{i=1}^{k+1} (-1)^i (F\psi(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_{k+1})) \circ a_i$$

$$= d_{rsym}(F\psi)(a_0, a_1, \ldots, a_{k+1}).$$

Thus, we obtain an equivalence of the cochain complexes $\oplus_{k>0} C^k_{lie}(A, C^1(A, M))$ and $\oplus_{k>1} C^k_{rsym}(A, M)$. In particular, we have the isomorphism (3).

Since, $H^1_{rsym}(A, M) = Z^1_{rsym}(A, M)/B^1_{rsym}(A, M)$ and

$$Z^1_{rsym}(A, M) = \{\psi \in C^1(A, M) : d_{rsym}\psi = 0\} = Z^0_{lie}(A, C^1(A, M)),$$

$$B^1_{rsym}(A, M) = \{d_{rsym}m : m \in M, (m, a, b) = 0, \forall a, b \in A\} = M^{l.ass}/M^{l.ass} \cap M^{inv},$$

we have the exactness of (4). □

Definition. Let $f : C^*_{rsym}(A, M) \to C^*_{lie}(A, M)$ be a linear operator such that

$$f : C^k_{rsym}(A, M) \to C^k_{lie}(A, M),$$

$$f\psi(a_1, \ldots, a_k) = \sum_{i=1}^{k} (-1)^{i+k+1} \psi(a_i, a_1, \ldots, \hat{a}_i, \ldots, a_k).$$

We introduce the subspaces

$$\bar{C}^*_{rsym}(A, M) = \oplus_k \bar{C}^k_{rsym}(A, M),$$

$$\bar{C}^k_{rsym}(A, M) = \{\psi \in C^{k+2}_{rsym}(A, M) : f\psi = 0\},\ k \ge 0.$$

Theorem 3.5. *Let A be a right-symmetric algebra and M be an A-module. Then the operator $f : C^*_{rsym}(A, M) \to C^*_{lie}(A, M)$ is a homomorphism of cochain complexes and the following cohomological sequence is exact:*

$$0 \to Z^1_{rsym}(A, M) \to Z^1_{lie}(A, M) \xrightarrow{\delta}$$

$$\bar{H}^0_{rsym}(A, M) \to H^2_{rsym}(A, M) \to H^2_{lie}(A, M) \xrightarrow{\delta} \bar{H}^1_{rsym}(A, M) \to \cdots$$

$$\xrightarrow{\delta} \bar{H}^{k-2}_{rsym}(A, M) \to H^k_{rsym}(A, M) \to H^k_{lie}(A, M) \xrightarrow{\delta} \bar{H}^{k-1}_{rsym}(A, M) \to \cdots$$

A connecting homomorphism

$$\delta : \bar{H}^k_{lie}(A, M) \to H^{k-1}_{rsym}(A, M)$$

is induced by a homomorphism

$$\delta : \bar{Z}^k_{lie}(A, M) \to Z^{k-1}_{rsym}(A, M), \quad \psi \mapsto d_{rsym}\psi.$$

Proof. We will check that for $k > 0$ the following diagram is commutative:

$$\begin{array}{ccc} C^k_{rsym}(A, M) & \xrightarrow{d_{rsym}} & C^{k+1}_{rsym}(A, M) \\ \downarrow f & & \downarrow f \\ C^k_{lie}(A, M)) & \xrightarrow{d_{lie}} & C^{k+1}_{lie}(A, M) \end{array}$$

For $\psi \in C^k_{rsym}(A, M)$, we have

$$\begin{aligned}
& f d_{rsym}\psi(a_1, \ldots, a_{k+1}) \\
&= \sum_s (-1)^{s+k} d_{rsym}\psi(a_s, a_1, \ldots, \hat{a}_s, \ldots, a_{k+1}) \\
&= \sum_{i<s} (-1)^{i+s+k+1} a_s \circ (\psi(a_i, a_1, \underline{\ldots, \hat{a}_i}, \ldots, \hat{a}_s, \ldots, a_k)) \\
&\quad + \sum_{s<i} (-1)^{i+s+k} a_s \circ \psi(a_i, a_1, \ldots, \underline{\underline{\hat{a}_s, \ldots}}, \hat{a}_i, \ldots, a_{k+1}) \\
&\quad - \sum_{i<s} (-1)^{i+s+k+1} \psi(a_s \circ a_i, a_1, \ldots, \utilde{\hat{a}_i, \ldots}, \hat{a}_s, \ldots, a_k)) \\
&\quad - \sum_{s<i} (-1)^{i+s+k} \psi(a_s \circ a_i, a_1, \ldots, \utilde{\hat{a}_s, \ldots}, \hat{a}_i \ldots, a_{k+1}) \\
&\quad + \sum_{i<j<s} (-1)^{s+i+k} \psi(a_s, a_1, \ldots, \hat{a}_i, \ldots, \utilde{a_{j-1},} [a_i, a_j], \ldots, \hat{a}_s, \ldots, a_{k+1}) \\
&\quad + \sum_{i<s<j} (-1)^{s+i+k} \psi(a_s, a_1, \ldots, \hat{a}_i, \ldots, \utilde{\hat{a}_s, \ldots}, a_{j-1}, [a_i, a_j], \ldots, a_{k+1}) \\
&\quad + \sum_{s<i<j} (-1)^{s+i+k+1} \psi(a_s, a_1, \ldots, \hat{a}_s, \ldots, \underline{\underline{\hat{a}_i, \ldots}}, a_{j-1}, [a_i, a_j], \ldots, a_{k+1}) \\
&\quad - \sum_{i<s} (-1)^{i+s+k+1} \psi(a_s, a_1, \ldots, \underline{\hat{a}_i, \ldots}, \hat{a}_s, \ldots, a_k)) \circ a_i \\
&\quad - \sum_{s<i} (-1)^{i+s+k} \psi(a_s, a_1, \ldots, \underline{\underline{\hat{a}_s, \ldots}}, \hat{a}_i \ldots, a_{k+1}) \circ a_i
\end{aligned}$$

and

$$\begin{aligned}
& d_{lie} f\psi(a_1, \ldots, a_{k+1}) \\
&= \sum_{i<j} (-1)^i f\psi(a_1, \ldots, \hat{a}_i, \ldots, a_{j-1}, [a_i, a_j], \ldots, a_{k+1}) \\
&\quad + \sum_i (-1)^i [f\psi(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}), a_i] \\
&= \sum_{s<i<j} (-1)^{i+s+k+1} \psi(a_s, a_1, \ldots, \hat{a}_s, \ldots, \underline{\underline{\hat{a}_i, \ldots}}, a_{j-1}, [a_i, a_j], \ldots, a_{k+1}) \\
&\quad + \sum_{i<s<j} (-1)^{i+k+s} \psi(a_s, a_1, \ldots, \hat{a}_i, \ldots, \utilde{\hat{a}_s, \ldots}, a_{j-1}, [a_i, a_j], \ldots, a_{k+1})
\end{aligned}$$

$$+(-1)^{i+j+k}\psi([a_i,a_j],a_1,\dots,\underset{\sim\sim\sim}{\hat{a}_i},\dots,\hat{a}_j,\dots,a_{k+1})$$

$$+\sum_{i<j<s}(-1)^{i+k+s}\psi(a_s,a_1,\dots,\hat{a}_i,\dots,\underset{\sim\sim\sim}{a_{j-1},[a_i,a_j]},\dots,\hat{a}_s,\dots,a_{k+1})$$

$$+\sum_{s<i}(-1)^{i+s+k}[\psi(a_s,a_1,\dots,\underline{\underline{\hat{a}_s,\dots,}}\hat{a}_i,\dots,a_{k+1}),a_i]$$

$$+\sum_{i<s}(-1)^{i+s+k+1}[\psi(a_s,a_1,\dots,\underline{\hat{a}_i,\dots,}\hat{a}_s,\dots,a_{k+1}),a_i].$$

Thus, according to the right-symmetric identity,

$$fd_{rsym}\psi=d_{lie}f\psi,\quad \forall\psi\in C^k_{rsym}(A,M),\quad \forall k>0.$$

So, we have a short exact sequence of cochain complexes

$$0\to\oplus_{k>0}\bar{C}^k_{rsym}(A,M)\to\oplus_{k>0}C^k_{rsym}(A,M)\to\oplus_{k>0}C^k_{lie}(A,M)\to 0.$$

In particul, the long cohomological sequence

$$\bar{H}^0_{rsym}(A,M)\to H^2_{rsym}(A,M)\to H^2_{lie}(A,M)\to\cdots$$

$$\to\bar{H}^{k-2}_{rsym}(A,M)\to H^k_{rsym}(A,M)\to H^k_{lie}(A,M)\to\cdots$$

is exact. The exactness of the beginning part

$$0\to Z^1_{rsym}(A,M)\to Z^1_{lie}(A,M)\to\bar{H}^0_{rsym}(A,M)\to H^2_{rsym}(A,M)$$

will be checked directly. It is clear that $d_{lie}\psi=0$ if $d_{rsym}\psi=0,\ \psi\in C^1(A,M)$. So, the natural homomorphism $Z^1_{rsym}(A,M)\to Z^1_{lie}(A,M)$ is a monomorphism. Let $\delta\phi,\ \phi\in Z^1_{lie}(A,M)$, give us a trivial class in $\bar{H}^0(A,M)=\bar{Z}^0_{rsym}(A,M)$. Then $\phi\in Z^1_{rsym}(A,M)$, following from $\delta\phi=d_{rsym}\phi$. Suppose that $\sigma\in\bar{Z}^0_{rsym}(A,M)$ is a coboundary in $Z^2_{rsym}(A,M)$, say $\sigma=d_{rsym}\omega$, for some $\omega\in C^1_{rsym}(A,M)$. Then $d_{rsym}\omega(a,b)=\sigma(a,b)=\sigma(b,a)=d_{rsym}\omega(b,a)$ for any $a,b\in A$. This means that $d_{lie}\omega=0$.

The theorem is completely proved. □

3.5. Cup product in right-symmetric cohomologies

Theorem 3.6. *Assume that the cup product of A-modules $\cup: M\times N\to S$ is given. Then the bilinear map*

$$C^{*+1}_{rsym}(A,M)\times C^*_{lie}(A,N)\to C^{*+1}_{rsym}(A,S),\ (\psi,\phi)\mapsto\psi\cup\phi,$$

defined by

$$C^{k+1}_{rsym}(A,M)\times C^l_{lie}(A,N)\to C^{k+l+1}_{rsym}(A,S),\ (\psi,\phi)\mapsto\psi\cup\phi,$$

$$\psi\cup\phi(a_0,a_1,\dots,a_{k+l})$$

$$=\sum_{\substack{\sigma\in Sym_{k+l},\\ \sigma(1)<\dots<\sigma(k),\\ \sigma(k+1)<\dots<\sigma(k+l)}} sgn\,\sigma\,\psi(a_0,a_{\sigma(1)},\dots,a_{\sigma(k)})\cup\phi(a_{\sigma(k+1)},\dots,a_{\sigma(k+l)}).$$

is also a cup product:

$$(\alpha\cup\beta)\rho_{rsym}(x)=\alpha\rho_{rsym}(x)\cup\beta+\alpha\cup\beta\rho_{lie}(x),\ \ \forall\alpha\in C^{*+1}_{rsym}(A,M),\ \forall\beta\in C^*_{lie}(A,N).$$

Moreover,

$$d_{rsym}(\psi\cup\phi)=d_{rsym}\psi\cup\phi-(-1)^k\psi\cup d_{lie}\phi,\tag{5}$$

for any $\psi\in C^{k+1}_{rsym}(A,M),\ \phi\in C^l_{lie}(A,N),\ \ k,l\geq 0.$

Proof. By Theorem 3.4,

$$F(\eta \cup \phi) = F(\eta) \cup \phi,$$
$$F(\alpha \rho_{lie}(x)) = (F\alpha)\rho_{rsym}(x),$$

for any $\eta \in C^k_{lie}(A, C^1(A, M))$, $\alpha \in C^{k+1}_{rsym}(A, M)$, $\phi \in C^l_{lie}(A, N)$, $x \in A$.

The cup product $M \times N \to S$, $(m, n) \mapsto m \cup n$, of A-modules extends a cup product of A^{lie}-modules:

$$C^1(A, M)^{lie} \times N^{lie} \to C^1(A, S)^{lie},$$
$$(f, n) \mapsto f \cup n, \quad (f \cup n)(a) = f(a) \cup n.$$

We check the correctness of this definition:

$$((f \cup n) \circ (a))(b)$$
$$= d_{rsym}(f \cup n)(b, a)$$
$$= b \circ ((f \cup n)(a)) - (f \cup n)(b \circ a) + ((f \cup n)(b)) \circ a$$
$$= b \circ (f(a) \cup n) - f(b \circ a) \cup n + (f(b) \cup n) \circ a$$
$$= (b \circ f(a)) \cup n - f(b \circ a) \cup n + (f(b) \circ a) \cup n + f(b) \cup [n, a]$$
$$= (d_{rsym} f(b, a)) \cup n + f(b) \cup [n, a]$$
$$= (f \circ a)(b) \cup n + f(b) \cup [n, a]$$
$$= ((f \circ a) \cup n + (f \cup [n, a]))(b).$$

Thus, we have the cup product of the Chevalley–Eilenberg cochain complexes [16]:

$$C^k_{lie}(A, C^1(A, M)) \times C^l_{lie}(A, N) \to C^{k+l}_{lie}(A, C^1(A, S))$$

$$\{(\eta \cup \phi)(a_1, \dots, a_{k+l})\}(a_0)$$
$$= \sum_{\substack{\sigma \in Sym_{k+l},\\ \sigma(1) < \cdots < \sigma(k),\\ \sigma(k+1) < \cdots < \sigma(k+l)}} sgn\, \sigma \, \{\eta(a_{\sigma(1)}, \dots, a_{\sigma(k)}) \cup \phi(a_{\sigma(k+1)}, \dots, a_{\sigma(k+l)})\}(a_0).$$

We see that the cup products for Chevalley–Eilenberg complexes and right-symmetric complexes are compatible. Namely,

$$(F\eta) \cup \phi = F(\eta \cup \phi), \tag{6}$$

for any $\eta \in C^k_{lie}(A, C^1(A, M))$, $\phi \in C^l_{lie}(A, N)$. (For the definition of the isomorphism $F : C^k_{lie}(A, C^1(A, M)) \to C^{k+1}_{rsym}(A, M)$, see (2)). Since, according to [16], we have

$$d_{lie}(\eta \cup \phi) = d_{lie}\eta \cup \phi + (-1)^k \eta \cup d_{lie}\phi,$$

then according to (6), we have

$$d_{rsym}((F\eta) \cup \phi) = d_{rsym} F(\eta \cup \phi)$$
$$= F d_{lie}(\eta \cup \phi) = F(d_{lie}\eta \cup \phi + (-1)^k \eta \cup d_{lie}\phi)$$
$$= F d_{lie}\eta \cup \phi + (-1)^k F\eta \cup d_{lie}\phi$$
$$= d_{rsym} F\eta \cup \phi + (-1)^k F\eta \cup d_{lie}\phi.$$

By Theorem 3.4, for any $\psi \in C^{k+1}_{rsym}(A, M), k \geq 0$, there exist $\eta \in C^k(A, C^1(A, M))$ such that $\psi = F\eta$. Hence, (5) is true. □

Corollary 3.7. *The cup product*

$$C^{*+1}_{rsym}(A,M)\times C^{*}_{lie}(A,N)\to C^{*+1}_{rsym}(A,S),\quad (\psi,\phi)\mapsto \psi\cup\phi,$$

induces a cup product of the cohomology spaces

$$H^{k+1}_{rsym}(A,M)\times H^{l}_{lie}(A,N)\to H^{k+l+1}_{rsym}(A,S),\quad ([\psi],[\phi])\mapsto[\psi\cup\phi],\quad k>0,l\geq 0.$$

$$Z^{1}_{rsym}(A,M)\times H^{l}_{lie}(A,N)\to H^{l+1}_{rsym}(A,S),\quad (\psi,[\phi])\mapsto[\psi\cup\phi],\ l\geq 0.$$

Proof.

$$Z^{k+1}_{rsym}(A,M)\cup Z^{l}_{lie}(A,N)\subseteq Z^{k+l+1}_{rsym}(A,S),\quad k,l\geq 0,$$

$$B^{k+1}_{rsym}(A,M)\cup Z^{l}_{lie}(A,N)\subseteq B^{k+l+1}_{rsym}(A,S),\quad k>0,\ l\geq 0,$$

$$Z^{k+1}_{rsym}(A,N)\cup B^{l}_{lie}(A,N)\subseteq B^{k+l+1}_{rsym}(A,S),\quad k,l\geq 0.\quad \square$$

Note that for any module M of a right-symmetric algebra A and a trivial A-module $\mathcal{K}$ there exists a natural cup product:

$$M\times\mathcal{K}\to M,\quad (m,\lambda)\mapsto m\lambda.$$

So, we have a pairing of the cohomology spaces:

$$H^{*}_{rsym}(A,M)\times H^{*}_{lie}(A,\mathcal{K})\to H^{*}_{rsym}(A,M).$$

In particular, $H^{*}_{rsym}(A,M)$ has the natural structure of a module over $H^{*}_{lie}(A,\mathcal{K})$. As it turned out, in some cases $H^{*}_{rsym}(A,M)$ is a free $H^{*}_{lie}(A,\mathcal{K})$-module. In Sec. 5, we will see that this is the case if $A=gl_n^{rsym}$.

Denote by $\bar{M}$ an antisymmetric A-module obtained from M by $\bar{r}_a=r_a-l_a,\bar{l}_a=0$. One can construct another cup product:

$$\mathcal{K}\times M\to\bar{M},\quad \lambda\cup m=\lambda m.$$

We use this cup product in the consideration of the right-symmetric cohomology for $A=W_n^{rsym}$ (Sec. 5).

4. Deformations of Right-Symmetric Algebras

4.1. Deformation equations

We will follow the Gerstenhaber theory of deformations of algebras [14]. Let A be a right-symmetry algebra over a field $\mathcal{K}$ of any characteristic p. Let $\mathcal{K}((t))$ be the field of fractions for the formal power series algebra $\mathcal{K}[[x]]$. We extend the main field $\mathcal{K}$ until $\mathcal{K}((t))$ and construct on the vector space $A\otimes\mathcal{K}((t))$ a new right-symmetric multiplication

$$\mu_t=\mu_0+t\mu_1+t^2\mu_2+\dots,$$

where

$$\mu_i\in C^2_{rsym}(A,M),\quad i=0,1,2,\dots,\ \text{and}\ \mu_0(a,b)=a\circ b.$$

The right-symmetric condition for μ_t in terms of μ_k can be regarded as the following *deformation equations*:

$$\mu_1\in Z^2_{rsym}(A,A), \tag{DFR.1}$$

$$\sum_{l=1}^{k-1}\mu_l\star\mu_{k-l}=-d_{rsym}\mu_k, \tag{DFR.k}$$

$$k=2,3,\dots,$$

where

$$(\psi\star\phi)(a,b,c)=\psi(a,\phi(b,c))-\psi(\phi(a,b),c)-\psi(a,\phi(c,b))+\psi(\phi(a,c),b),$$

$$\psi, \phi \in C^2_{rsym}(A, M)$$

The right-symmetric deformations μ_t, and ν_t are said to be *equivalent* if there exists a map

$$g_t = g_0 + tg_1 + tg_2 + \cdots, \quad g_k \in C^1_{rsym}(A, A), \quad k = 0, 1, 2, \ldots,$$

with the identity map g_0, such that

$$g_t^{-1}(\mu_t(g_t(a), g_t(b))) = \nu_t(a, b), \ \forall a, b \in A.$$

In particular, for the equivalent deformations μ_t, ν_t, one should have

$$\nu_1 = \mu_1 + d_{rsym} g_1.$$

In other words, the first deformation terms, the so-called *local deformations*, will define equivalent 2-right-symmetry cohomology classes $[\mu_1] = [\nu_1]$.

Conversely, suppose that there is given a 2-cocycle of a right-symmetric algebra with coefficients in the regular module, $\psi \in Z^2_{rsym}(A, A)$, with a cohomology class $[\psi] \in H^2_{rsym}(A, A)$. One can take $\mu_1 := \psi$, and try to construct μ_k that satisfy deformation equations. Obviously, (DFR.1) is true. We will say that a *local deformation* $\mu_1 = \psi$ *can be extended to a global deformation until k-th term,* if there exist $\mu_2, \ldots, \mu_k$, such that equations (DFR.k) are true. If this is the case for any $k > 0$, we will say that a *local deformation* μ_1 *can be extended to a global deformation* μ_t or, equivalently, that μ_t is a global deformation or an extension of μ_1. Set

$$Obs_k(\psi) = \sum_{l=1}^{k-1} \mu_l \star \mu_{k-l}.$$

Note that the definition of $Obs_k(\psi)$ depends not only on ψ but also on the first $k-1$ terms of the deformation.

4.2. Third cohomologies as obstructions

Proposition 4.1. *Suppose that a local deformation $\mu_1 = \psi$ can be extended to a global deformation to the $(k-1)$th term. Then, $Obs_k(\psi) \in Z^3_{rsym}(A, A)$ and an extension of μ_1 to the kth term is possible if and only if $[Obs_k(\psi)] = 0$.*

Proof. For $\alpha \in C^{k+1}(A, A), \beta \in C^{l+1}(A, A)$, define multiplications $\alpha * \beta \in C^{k+l+1}(A, A)$, $\alpha \smile \beta \in C^{k+l+2}(A, A)$ by

$$\alpha * \beta(a_1, \ldots, a_{k+l+1})$$
$$= \sum_{s=1}^{k+1} (-1)^{(s+1)l} \alpha(a_1, \ldots, a_{s-1}, \beta(a_{s+1}, \ldots, a_{s+l}), a_{s+l+1}, \ldots, a_{k+l+1}).$$

$$\alpha \smile \beta(a_1, \ldots, a_{k+l+2})$$
$$= \alpha(a_1, \ldots, a_{k+1}) \circ \beta(a_{k+2}, \ldots, a_{k+l+2}).$$

Then,

$$\psi \star \phi(a, b, c) = \psi * \phi(a, c, b) - \psi * \phi(a, b, c), \quad \psi, \phi \in C^2(A, A),$$

and

$$d_{rsym} Obs_k(\psi)(a_0, a_1, a_2, a_3)$$
$$= \sum_{l+s=k, l>0, s>0} \sum_{\sigma \in Sym_3} sgn\, \sigma\, d_{ass}(\mu_l * \mu_s)(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

where d_{ass} stands for the Hochschild coboundary operator as in associative algebras. By [15], § 7, Theorem 3,

$$d_{ass} \alpha * \beta = \alpha * d_{ass} \beta - d_{ass} \alpha * \beta - \alpha \smile \beta + \beta \smile \alpha, \quad \alpha, \beta \in C^2(A, A).$$

Note that

$$\sum_{l+s=k,l>0,s>0} \mu_l \smile \mu_s - \mu_s \smile \mu_l = \sum_{l+s-k,l>0,s>0} \mu_l \smile \mu_s - \sum_{l+s-k,l>0,s>0} \mu_l \smile \mu_s = 0.$$

Hence, according to conditions (DFR.l), $l < k$,

$$d_{rsym}Obs_k(\psi)(a_0, a_1, a_2, a_3)$$

$$= \sum_{l+s=k,l>0,s>0} \sum_{\sigma \in Sym_3} sgn\, \sigma\, d_{ass}(\mu_l \smile \mu_s)(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

$$= \sum_{l+s=k,l>0,s>0} \sum_{\sigma \in Sym_3} sgn\, \sigma\, \mu_l * d_{ass}\mu_s(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

$$-sgn\, \sigma\, d_{ass}\mu_l * \mu_s(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

$$= \sum_{l+s=k,l>0,s>0} \sum_{\sigma \in Sym_3} sgn\, \sigma\, \mu_l(d_{ass}\mu_s(a_0, a_{\sigma(1)}, a_{\sigma(2)}), a_{\sigma(3)})$$

$$-sgn\, \sigma\, \mu_l(a_0, d_{ass}\mu_s(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}))$$

$$-sgn\, \sigma\, d_{ass}\mu_l(\mu_s(a_0, a_{\sigma(1)}), a_{\sigma(2)}, a_{\sigma(3)})$$

$$+sgn\, \sigma\, d_{ass}\mu_l(a_0, \mu_s(a_{\sigma(1)}, a_{\sigma(2)}), a_{\sigma(3)})$$

$$-sgn\, \sigma\, d_{ass}\mu_l(a_0, a_{\sigma(1)}, \mu_s(a_{\sigma(2)}, a_{\sigma(3)}))$$

$$= \sum_{l+s=k,l>0,s>0}$$

$$\{\mu_l(d_{rsym}\mu_s(a_0, a_1, a_2), a_3) - \mu_l(d_{rsym}\mu_s(a_0, a_1, a_3), a_2) + \mu_l(d_{rsym}\mu_s(a_0, a_2, a_3), a_1)$$

$$-\mu_l(a_0, d_{rsym}\mu_s(a_1, a_2, a_3)) + \mu_l(a_0, d_{rsym}\mu_s(a_2, a_1, a_3)) - \mu_l(a_0, d_{rsym}\mu_s(a_3, a_1, a_2))$$

$$-d_{rsym}\mu_l(\mu_s(a_0, a_1), a_2, a_3) + d_{rsym}\mu_l(\mu_s(a_0, a_2), a_1, a_3) - d_{rsym}\mu_l(\mu_s(a_0, a_3), a_1, a_2)$$

$$+d_{rsym}\mu_l(a_0, \mu_s(a_1, a_2), a_3) - d_{rsym}\mu_l(a_0, \mu_s(a_2, a_1), a_3)$$

$$-d_{rsym}\mu_l(a_0, \mu_s(a_1, a_3), a_2) + d_{rsym}\mu_l(a_0, \mu_s(a_3, a_1), a_2)$$

$$+d_{rsym}\mu_l(a_0, \mu_s(a_2, a_3), a_1) - d_{rsym}\mu_l(a_0, \mu_s(a_3, a_2), a_1)\}$$

$$= S_1 + S_2,$$

where

$$S_1 = \sum_{l+s=k,l>0,s>0} \sum_{s_1+s_2=s,s_1,s_2>0}$$

$$\{-\mu_l(\mu_{s_1} \star \mu_{s_2}(a_0, a_1, a_2), a_3) + \mu_l(\mu_{s_1} \star \mu_{s_2}(a_0, a_1, a_3), a_2) - \mu_l(\mu_{s_1} \star \mu_{s_2}(a_0, a_2, a_3), a_1)$$

$$+\mu_l(a_0, \mu_{s_1} \star \mu_{s_2}(a_1, a_2, a_3)) - \mu_l(a_0, \mu_{s_1} \star \mu_{s_2}(a_2, a_1, a_3)) + \mu_l(a_0, \mu_{s_1} \star \mu_{s_2}(a_3, a_1, a_2))\},$$

$$S_2 = \sum_{l+s=k,l>0,s>0} \sum_{l_1+l_2=l,l_1,l_2>0}$$

$$\{\mu_{l_1} \star \mu_{l_2}(\mu_s(a_0, a_1), a_2, a_3) - \mu_{l_1} \star \mu_{l_2}(\mu_s(a_0, a_2), a_1, a_3) + \mu_{l_1} \star \mu_{l_2}(\mu_s(a_0, a_3), a_1, a_2)$$

$$-\mu_{l_1} \star \mu_{l_2}(a_0, \mu_s(a_1, a_2), a_3) + \mu_{l_1} \star \mu_{l_2}(a_0, \mu_s(a_2, a_1), a_3)$$

$$+\mu_{l_1} \star \mu_{l_2}(a_0, \mu_s(a_1, a_3), a_2) - \mu_{l_1} \star \mu_{l_2}(a_0, \mu_s(a_3, a_1), a_2)$$

$$-\mu_{l_1} \star \mu_{l_2}(a_0, \mu_s(a_2, a_3), a_1) + \mu_{l_1} \star \mu_{l_2}(a_0, \mu_s(a_3, a_2), a_1)\}.$$

We have

$$S_1 = \sum_{l+s=k,l>0,s>0} \sum_{s_1+s_2=s,s_1>0,s_2>0}$$

$$\{-\mu_l(\mu_{s_1}*\mu_{s_2}(a_0,a_1,a_2),a_3)+\mu_l(\mu_{s_1}*\mu_{s_2}(a_0,a_1,a_3),a_2)-\mu_l(\mu_{s_1}*\mu_{s_2}(a_0,a_2,a_3),a_1)$$

$$+\mu_l(a_0,\mu_{s_1}*\mu_{s_2}(a_1,a_2,a_3))-\mu_l(a_0,\mu_{s_1}*\mu_{s_2}(a_2,a_1,a_3))+\mu_l(a_0,\mu_{s_1}*\mu_{s_2}(a_3,a_1,a_2))$$

$$+\mu_l(\mu_{s_1}*\mu_{s_2}(a_0,a_2,a_1),a_3)-\mu_l(\mu_{s_1}*\mu_{s_2}(a_0,a_3,a_1),a_2)+\mu_l(\mu_{s_1}*\mu_{s_2}(a_0,a_3,a_2),a_1)$$

$$-\mu_l(a_0,\mu_{s_1}*\mu_{s_2}(a_1,a_3,a_2))+\mu_l(a_0,\mu_{s_1}*\mu_{s_2}(a_2,a_3,a_1))-\mu_l(a_0,\mu_{s_1}*\mu_{s_2}(a_3,a_2,a_1))\}$$

$$=\sum_{l+s_1+s_2=k,l>0,s_1>0,s_2>0}\ \sum_{\sigma\in Sym_3}-sgn\,\sigma\,\mu_l*(\mu_{s_1}*\mu_{s_2})(a_0,a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}),$$

and

$$S_2=\sum_{l+s=k,l>0,s>0}\ \sum_{l_1+l_2=l,l_1>0,l_2>0}$$

$$\{\mu_{l_1}*\mu_{l_2}(\mu_s(a_0,a_1),a_2,a_3)-\mu_{l_1}*\mu_{l_2}(\mu_s(a_0,a_2),a_1,a_3)+\mu_{l_1}*\mu_{l_2}(\mu_s(a_0,a_3),a_1,a_2)$$

$$-\mu_{l_1}*\mu_{l_2}(a_0,\mu_s(a_1,a_2),a_3)+\mu_{l_1}*\mu_{l_2}(a_0,\mu_s(a_2,a_1),a_3)$$

$$+\mu_{l_1}*\mu_{l_2}(a_0,\mu_s(a_1,a_3),a_2)-\mu_{l_1}*\mu_{l_2}(a_0,\mu_s(a_3,a_1),a_2)$$

$$-\mu_{l_1}*\mu_{l_2}(a_0,\mu_s(a_2,a_3),a_1)+\mu_{l_1}*\mu_{l_2}(a_0,\mu_s(a_3,a_2),a_1)$$

$$-\mu_{l_1}*\mu_{l_2}(\mu_s(a_0,a_1),a_3,a_2)+\mu_{l_1}*\mu_{l_2}(\mu_s(a_0,a_2),a_3,a_1)-\mu_{l_1}*\mu_{l_2}(\mu_s(a_0,a_3),a_2,a_1)$$

$$+\mu_{l_1}*\mu_{l_2}(a_0,a_3,\mu_s(a_1,a_2))-\mu_{l_1}*\mu_{l_2}(a_0,a_3,\mu_s(a_2,a_1))$$

$$-\mu_{l_1}*\mu_{l_2}(a_0,a_2,\mu_s(a_1,a_3))+\mu_{l_1}*\mu_{l_2}(a_0,a_2,\mu_s(a_3,a_1))$$

$$+\mu_{l_1}*\mu_{l_2}(a_0,a_1,\mu_s(a_2,a_3))-\mu_{l_1}*\mu_{l_2}(a_0,a_1,\mu_s(a_3,a_2))\}$$

$$=\sum_{l_1+l_2+s=k,l_1>0,l_2>0,s>0} sgn\,\sigma\,(\mu_{l_1}*\mu_{l_2})*\mu_s(a_0,a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}),$$

Let $\alpha,\beta,\gamma\in C^2(A,A)$. Then,

$$\{\alpha*(\beta*\gamma)-(\alpha*\beta)*\gamma\}(a,b,c,d)$$

$$=\alpha(\beta*\gamma(a,b,c),d)+\alpha(a,\beta*\gamma(b,c,d))$$

$$-\alpha*\beta(\gamma(a,b),c,d)+\alpha*\beta(a,\gamma(b,c),d)-\alpha*\beta(a,b,\gamma(c,d))$$

$$=\alpha(\beta(\gamma(a,b),c),d)-\alpha(\beta(a,\gamma(b,c)),d)$$

$$+\alpha(a,\beta(\gamma(b,c),d))-\alpha(a,\beta(b,\gamma(c,d)))$$

$$-\alpha(\beta(\gamma(a,b),c),d)+\alpha(\gamma(a,b),\beta(c,d))$$

$$+\alpha(\beta(a,\gamma(b,c)),d)-\alpha(a,\beta(\gamma(b,c),d))$$

$$-\alpha(\beta(a,b),\gamma(c,d))+\alpha(a,\beta(b,\gamma(c,d)))=$$

$$-\alpha(\beta(a,b),\gamma(c,d))+\alpha(\gamma(a,b),\beta(c,d)).$$

So, for any $\alpha, \beta, \gamma \in C^2(A, A)$, we have

$$\alpha * (\beta * \gamma + \gamma * \beta) - (\alpha * \beta) * \gamma - (\alpha * \gamma) * \beta = 0$$

For these reasons,

$$S_1 = \sum_{s_1+s_2+s_3=k, s_1>0, s_2>0, s_3>0} \sum_{\sigma \in Sym_3} -sgn\,\sigma\, \mu_{s_1} * (\mu_{s_2} * \mu_{s_3})(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

$$= - \sum_{l_1+l_2+l_3=k, l_1>0, l_2>0, l_3>0} sgn\,\sigma\, (\mu_{l_1} * \mu_{l_2}) * \mu_{l_3}(a_0, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = -S_2.$$

So, we prove that $d_{rsym} Obs_k(\psi) = 0$ if $d_{rsym} Obs_l(\psi) = 0$, for any $0 < l < k$. □

Corollary 4.2. *If $H^3_{rsym}(A, A) = 0$, then any local deformation can be extended.*

4.3. Steenrod squares

Let $char\, k = p > 0$. In this subsection, we recall Gerstenhaber's construction [14] of a homomorphism with regard to right-symmetric algebras:

$$Sq : Z^1_{rsym}(A, A) \to Z^2_{rsym}(A, A), \quad D \mapsto SqD.$$

For any derivation $D \in Z^1_{rsym}(A, A)$, its pth power D^p is also a derivation, $D^p \in Z^1_{rsym}(A, A)$. The proof is based on the following property of binomial coefficients: an integer $\binom{p}{a}$ is divisible by p, if $0 < a < p$. Then,

$$D^p(a \circ b) - D^p(a) \circ b - a \circ D^p(b)$$

$$= \sum_{i=1}^{p-1} \binom{p}{i} D^i(a) \circ D^{p-i}(b) \equiv 0 (mod\, p).$$

In particular, we can consider integers $\frac{\binom{p}{i}}{p}$, $0 < i < p$, modulo p and introduce a 2-cocycle SqD with coefficients in the regular module:

$$Sq\, D(a, b) = \sum_{l=1}^{p-1} D^i(a) \circ D^{p-i}(b)/i!(p-i)!.$$

This cocycle is called a Steenrod square of derivation D and can be interpreterd as an obstruction to the extension of a derivation to an automorphism.

5. Calculations

5.1. Standard 2-cocycles of right-symmetric algebras

In this subsection, we will give the second interpretation of the identity $d^3_{rsym} m = 0$, $m \in M$, mentioned in Sec. 3.2.

Proposition 5.1. (i) *Let $\tilde{M}$ be a module over a right-symmetric algebra A and M its submodule. Suppose that for $\tilde{m} \in \tilde{M}$,*

$$(\tilde{m}, a, b) \in M, \quad \forall a, b \in A.$$

Then the 2-cochain $\psi_{\tilde{m}} \in C^2(A, M)$ defined by

$$\psi_{\tilde{m}}(a, b) = \tilde{m} \circ (a \circ b) - (\tilde{m} \circ a) \circ b$$

is a symmetric 2-cocycle, $\psi_{\tilde{m}} \in \bar{Z}^2(A, M)$.

If $\tilde{m} \circ a \in M$, $\forall a \in A$, then $[\psi_{\tilde{m}}] = [\phi_{\tilde{m}}]$, where

$$\phi_{\tilde{m}}(a, b) = a \circ (\tilde{m} \circ b).$$

Note that if we use the notation of Sec. 3.3, $\psi_{\tilde{m}} = \nabla(\tilde{m})$.

Proof.

$$\psi_{\tilde{m}}(a,b) = (m,a,b) = (m,b,a) = \psi_{\tilde{m}}(b,a),$$

$$d_{rsym}\psi_{\tilde{m}}(a,b,c)$$

$$= a\circ(\tilde{m},b,c) - a\circ(\tilde{m},c,b) - (\tilde{m},a\circ b,c) + (\tilde{m},a\circ c,b) + (\tilde{m},a,[b,c]) - (\tilde{m},a,b)\circ c + (\tilde{m},a,c)\circ b$$

$$-(\tilde{m},a\circ b,c) + (\tilde{m},a,b\circ c) - (\tilde{m},a,b)\circ c$$

$$+(\tilde{m},a\circ c,b) - (\tilde{m},a,c\circ b) + (\tilde{m},a,c)\circ b$$

$$= \tilde{m}\circ(a,b,c) - (\tilde{m}\circ a,b,c)$$

$$-\tilde{m}\circ(a,c,b) + (\tilde{m}\circ a,c,b) = 0.$$

If $\tilde{m}\circ a\in M$, then we can introduce a linear map $\omega : A\to M,\ a\mapsto \tilde{m}\circ a$. We obtain

$$\psi_{\tilde{m}}(a,b) + d_{rsym}\omega(a,b)$$

$$= \psi_{\tilde{m}}(a,b) - \omega(a\circ b) + \omega(a)\circ b + a\circ\omega(b)$$

$$= \psi_{\tilde{m}}(a,b) - \tilde{m}\circ(a\circ b) + (\tilde{m}\circ a)\circ b + a\circ(\tilde{m}\circ b)$$

$$= a\circ(\tilde{m}\circ b).$$

In other words, $\phi_{\tilde{m}} = \psi_{\tilde{m}} + d_{rsym}\omega$. □

5.2. Semi-center and derivations of W_n^{rsym}

Suppose that A is an algebra with multiplications $(a,b)\mapsto a\circ b$ and $(a,b)\mapsto a*b$ such that the following conditions hold:

$$a\circ(b\circ c) - (a\circ b)\circ c - a\circ(c\circ b) + (a\circ c)\circ b = 0, \tag{7}$$

$$a*(b*c) - b*(a*c) = 0, \tag{8}$$

$$a\circ(b*c) - b*(a\circ c) = 0, \tag{9}$$

$$(a*b - b*a - a\circ b - b\circ a)*c = 0, \tag{10}$$

$$(a\circ b - b\circ a)*c + a*(c\circ b) - (a*c)\circ b - b*(c\circ a) + (b*c)\circ a = 0. \tag{11}$$

In particular, A is a right-symmetric algebra.

Let $Z_l(A)$ be the left center of A, $Q_l(A)$ is the space of left units, and $N_l(A) = Z_l(A)\oplus Q_l(A)$ is the left semi-center.

Theorem 5.2. *For $A = W_n^{rsym}$ if $p=0$, or $A = W_n(\mathbf{m})$ if $p>0$,*

$$Z_l(A) = \{\partial_i : i=1,\dots,n\}\oplus\delta(p>0)\{\partial_i^{p^{k_i}-1} : 0<k_i<m_i, i=1,\dots,n\}\cong\mathcal{K}^n\oplus\delta(p>0)\mathcal{K}^{m-n},$$

$$Q_l(A) = \{e = (1/n)\sum_{i=1}^n x_i\partial_i\}\cong\mathcal{K},$$

$$Z^1_{rsym}(A,A) = \{ad\,\partial_i, ad\,x_i\partial_j, \delta(p>o)\partial_i^{p^{k_i}} : i,j,=1,\dots,n, 0<k_i<m_i\}$$

$$\cong\mathcal{K}^n\oplus gl_n\oplus\delta(p>0)\mathcal{K}^{m-n}.$$

Proof. Any derivation of the right-symmetric algebra A induces a derivation of the Lie algebra A^{lie} :

$$d_{rsym}(a,b)=0, \forall a,b,c\in A \Rightarrow d_{lie}f(a,b)=d_{rsym}f(a,b)-d_{rsym}(b,a)=0, \ \forall a,b,c\in A.$$

The corresponding homomorphism $Z^1_{rsym}(A,A)\to Z^1_{lie}(A,A)$ is a monomorphism. It is known that all Lie derivations of W_n are inner, i.e., have the form $ad\, u\partial_i$, where $u\in U, i=1,\dots,n$. In the case of $W_n(\mathrm{m}), p>0$, we also have outer derivations $\partial_i^{p^{k_i}}, 0<k_i<m_i, i=1,\dots,m_i$. Since

$$ada(b\circ c)-ad\,a(b)\circ c-b\circ ad\,a(c)$$

$$=a\circ(b\circ c)-(b\circ c)\circ a-(a\circ b)\circ c+(b\circ a)\circ c-b\circ(a\circ c)+b\circ(c\circ a)$$

$$=a\circ(b\circ c)-(a\circ b)\circ c,$$

the Lie derivation $ad\,a: b\mapsto [a,b]:=a\circ b-b\circ a$ is a right-symmetric derivation if and only if

$$a\in A^{l.ass}.$$

It is easy to verify that

$$u\partial_i\circ(v\partial_j\circ w\partial_s)-(u\partial_i\circ v\partial_j)\circ w\partial_s=\partial_j\partial_s(u)vw\partial_i.$$

Therefore,

$$A^{l.ass}=\{u\partial_i:\partial_j\partial_s(u)=0,\forall i,j=1,\dots,n\}=\{\partial_i,x_i\partial_j:i,j=1,\dots,n\}.$$

In the case $p>0$, direct calculations show that $\partial_i^{p^{k_i}}\in Z^1_{rsym}(A,A)$. Other statements of the theorem are obvious.

5.3. Pairing of W_n^{rsym}-modules and cocycle constructions

Theorem 5.3. *Let $A=W_n$, $p=0$, or $A=W_n(\mathrm{m}), p>3$. The space $H^2_{rsym}(A,A), p=0$, has a basis consisting of cocycle classes of four types, $\psi^1_{s,l,r},\psi^2_{l,r},\psi^3_{s,l},\psi^4_l$, $s,l,r=1,\dots,n$, such that*

$$\psi^1_{s,l,r}(u\partial_i,v\partial_j)=\delta_{j,r}x_r^{-1}(\delta_{i,s}uv\partial_l-x_s\partial_l(u)v\partial_i),$$

$$\psi^2_{l,r}(u\partial_i,v\partial_j)=\delta_{j,r}x_r^{-1}\partial_l(u)v\partial_i,$$

$$\psi^3_{s,l}(u\partial_i,v\partial_j)=(\delta_{i,s}u\partial_j(v)\partial_l-x_s\partial_l(u)\partial_j(v)\partial_i),$$

$$\psi^4_l(u\partial_i,v\partial_j)=\partial_l(u)\partial_j(v)\partial_i.$$

In the case $p>3$, the space $H^2_{rsym}(A,A)$ has a basis whose elements are the cohomological classes of the following cocycles of the five types to follow:

$$\psi^1_{s,l,r}(u\partial_i,v\partial_j)=\delta_{j,r}x_r^{p^{m_r}-1}(\delta_{i,s}uv\partial_l-x_s\partial_l(u)v\partial_i),$$

$$\psi^2_{l,k_l,r}(u\partial_i,v\partial_j)=\delta_{j,r}x_r^{p^{m_r}-1}\partial_l^{p^{k_l}}(u)v\partial_i,$$

$$\psi^3_{s,l}(u\partial_i,v\partial_j)=(\delta_{i,s}u\partial_j(v)\partial_l-x_s\partial_l(u)\partial_j(v)\partial_i),$$

$$\psi^4_{l,k_l}(u\partial_i,v\partial_j)=\partial_l^{p^{k_l}}(u)\partial_j(v)\partial_i,$$

$$\psi^5_{l,k_l}=Sq\,\partial_l^{p^{k_l}},$$

where $s,l,r=1,\dots,n$, $0\le k_l<m_l$.

The proof is based on the following observations.

Suppose that an A-module M preserves the action of $N_l(A)$:

$$z \circ m = 0, \forall z \in Z_l(A), \quad e \circ m = m, \forall e \in Q_l(A),$$

for any $m \in M$. We define an operator

$$C^{k+1}_{rsym}(A, M) \to C^k_{rsym}(A, M),$$
$$i_0(a)\psi(a_1, \dots, a_k) = \psi(a_0, a_1, \dots, a_k),$$

and an operator

$$T : C^k_{rsym}(A, A) \to C^{k+1}_{rsym}(A, A),$$
$$T\psi(a_0, a_1, \dots, a_k) = \sum_{i=1}^{k}(-1)^{i+k} a_i * \psi(a_0, a_1, \dots, \hat{a}_i, \dots, a_k).$$

Then

$$Td_{rsym} = d_{rsym}T,$$

and for any $a \in N_l(A), \psi \in Z^{k+1}_{rsym}(A, A)$,

$$i_0(a)\psi \in Z^k_{lie}(A, A_{anti}).$$

We define a pairing of the regular A-module A and an antisymmetric A-module U:

$$A \times U \to A, \quad u\partial_i \cup v = uv\partial_i.$$

Note that

$$A_{anti} \cong U \otimes Z_l(A).$$

In particular, we have the pairing

$$A \times A_{anti} \to A, \quad u\partial_i \cup v\partial_j = uv\partial_i.$$

Therefore, we have the imbedding

$$Z^1_{rsym}(A, A) \times H^k_{lie}(A, U) \to H^{k+1}_{rsym}(A, A).$$

Note that four types of the cocycles mentioned above can be obtained from $Z^1_{rsym}(A, A)$ (see Sec. 5.2) and $H^1_{lie}(A, U)$ by pairing $\psi \cup \phi$, $\psi \in Z^1_{rsym}(A, A)$, $\phi \in H^1_{lie}(A, U)$. Recall that $H^1_{lie}(A, U)$ can be generated by the classes of cocycles $u\partial_i \mapsto ux_r^{-1}\delta_{i,r}$ and $u\partial_i \mapsto \partial_i(u)$.

Another interpretation of cocycles of types 1 and 2 can be given in terms of standard cocycles (see Sec. 5.1). For simplicity, we consider only the case $p = 0$. We have

$$\psi^1_{s,l,r} = d\omega_{s,l,r}, \text{for} \quad \omega_{s,l,r}(u\partial_i) = \ln x_r[x_s\partial_l, u\partial_i],$$
$$\psi^2_{l,r} = d\omega_{l,r}, \text{for} \quad \omega_{l,r}(u\partial_i) = \ln x_r\partial_l(u)\partial_i,$$

and

$$\nabla(x_s \ln x_r\partial_l)(u\partial_i, v\partial_j) = \partial_i\partial_j(x_s \ln x_r)uv,$$
$$\nabla(\ln x_r\partial_l)(u\partial_i, v\partial_j) = -\delta_{i,r}\delta_{j,r}uv\partial_l.$$

Therefore,

$$[\psi^1_{s,l,r}] = [\nabla(x_s \ln x_r\partial_l)],$$
$$[\psi^2_{l,r}] = [\nabla(\ln x_r\partial_l)],$$

since

$$\psi^1_{s,l,r} = \nabla(x_s \ln x_r\partial_l) - d^{rsym}\omega^1_{s,l,r}, \text{for} \ \ \omega^1_{s,l,r} \in C^1_{rsym}(W_n, W_n), \ \ \omega^1_{s,l,r}(u\partial_i) = \delta_{i,r}x_r^{-1}x_su\partial_l,$$

$$\psi^2_{l,r} = \nabla(\ln x_r\partial_l) - d^{rsym}\omega^2_{l,r}, \text{for} \ \ \omega^2_{l,r} \in C^1_{rsym}(W_n, W_n), \ \ \omega^2_{l,r}(u\partial_i) = \delta_{i,r}x_r^{-1}u\partial_l.$$

Theorem 5.4. *Let $A = W_1^{rsym}$ if $p = 0$, and $A = W_1(m)$ if $p > 3$. Then $H^2_{rsym}(A, A)$ has dimension 4 if $p = 0$, and cohomological classes of the following cocycles constitute a basis:*

$$\psi^1(u\partial, v\partial) = x^{-1}uv\partial,$$

$$\psi^2(u\partial, v\partial) = x^{-1}\partial(u)v\partial,$$

$$\psi^3(u\partial, v\partial) = (u - x\partial(u))\partial(v)\partial,$$

$$\psi^4(u\partial, v\partial) = \partial(u)\partial(v)\partial.$$

(Recall that $\partial = \partial_x$, for $n = 1$.)

For $A = W_1(m), p > 3$, the group $H^2_{rsym}(A, A)$ is $(3m + 2)$-dimensional and the classes of the following cocycles form a basis:

$$\psi^1(u\partial, v\partial) = x^{p^m-1}uv\partial,$$

$$\psi^2_k(u\partial, v\partial) = x^{p^m-1}\partial^{p^k}(u)v\partial, \quad 0 \le k < m,$$

$$\psi^3(u\partial, v\partial) = (u - x\partial(u))\partial(v)\partial,$$

$$\psi^4_k(u\partial, v\partial) = \partial^{p^k}(u)\partial(v)\partial, \quad 0 \le k < m.$$

$$Sq\,D : (a, b) \mapsto \sum_{i=1}^{p-1} D^i(a) \circ D^{p-i}(b)/(i!(p - i)!), \quad D = \partial^{p^k}, 0 \le k < m.$$

Cocycles of types 1 and 2 are also Novikov cocycles. A local deformation $\psi = \sum_{i=1}^4 t_i\psi^i$, $p = 0$, can be extended if and only if $t_1t_3 = 0$, $t_2t_4 = 0$.

5.4. Right-symmetric central extensions of Novikov algebras

Let A be a Novikov algebra, $R \in Der_0\,A := \{D \in Der\,A : a \circ R(b) = b \circ R(a),\ \forall a, b \in A\}$, and $\pi : A \to \mathcal{K}$ a linear map such that

$$\pi(R(a)) = 0, \quad \forall a \in A.$$

Define $\psi \in C^2_{rsym}(A, \mathcal{K})$ by

$$\vartheta(a, b) = \pi(a \circ R(b)).$$

Lemma 5.5. $\vartheta \in \bar{Z}^2_{rsym}(A, \mathcal{K})$.

Proof. Since $a \circ R(b) = b \circ R(a)$, it follows that

$$\vartheta(a, b) = \vartheta(b, a).$$

We have

$$d_{rsym}\vartheta(a, b, c)$$

$$= \pi(-\,(a \circ \underline{b) \circ R}(c) + (a \circ \underline{\underline{c) \circ R}}(b) + a \circ R[b, c])$$

$$= -(a \circ R(c)) \circ b - a \circ [b, R(c)]$$

$$+ (a \circ R(b)) \circ c + a[c, R(b)] + a \circ R[b, c])$$

$$= \pi(-(a \circ R(c)) \circ b + (a \circ R(b)) \circ c)$$

$$= \pi(-R((a \circ c) \circ b) + (R(a) \circ c) \circ b + (a \circ c) \circ R(b)$$

$$+ (a \circ R(b)) \circ c)$$

$$= \pi((R(a) \circ c) \circ b + b \circ R(a \circ c) + (a \circ R(b)) \circ c)$$

$$= \pi((R(a) \circ c) \circ b + b \circ (R(a) \circ c) + b \circ (a \circ R(c)) + (a \circ R(b)) \circ c)$$

$$= \pi((R(a) \circ c) \circ b + R(a) \circ (b \circ c) + a \circ (b \circ R(c)) + (a \circ R(b)) \circ c)$$

$$= \pi((R(a) \circ c) \circ b + R(a \circ (b \circ c)) - a \circ (R(b) \circ c) + (a \circ R(b)) \circ c)$$

$$= \pi((R(a) \circ c) \circ b - a \circ (c \circ R(b)) + (a \circ c) \circ R(b))$$

$$= \pi(-a \circ (c \circ R(b)) + R((a \circ c) \circ b) - (a \circ R(c)) \circ b)$$

$$= \pi(-a \circ (c \circ R(b)) - (a \circ R(c)) \circ b)$$

$$= \pi(-R(a \circ (c \circ b)) + R(a) \circ (c \circ b) + a \circ (R(c) \circ b) - (a \circ R(c)) \circ b)$$

$$= \pi(R(a) \circ (c \circ b) + a \circ (b \circ R(c)) - (a \circ b) \circ R(c))$$

$$= \pi(c \circ (R(a) \circ b) + a \circ (c \circ R(b)) - (a \circ b) \circ R(c))$$

$$= \pi(c \circ (R(a) \circ b) + c \circ (a \circ R(b)) - (a \circ b) \circ R(c))$$

$$= \pi(c \circ R(a \circ b) - (a \circ b) \circ R(c)) = 0.$$

So, $\vartheta \in \bar{Z}^2_{rsym}(A, \mathcal{K})$. □

In particular, the algebras

$$A = W_1^{rsym} = \{e_i : e_i \circ e_j = (i+1)e_{i+j}, \;\; i, j \in \mathbf{Z}\}, \; p = 0,$$

$$A = W_1^{rsym}(m) = \{e_i : e_i \circ e_j = \binom{i+j+1}{i} e_{i+j}, \; -1 \le i, j \le p^m - 1\}, \; p > 0,$$

have right-symmetric 2-cocycles with coefficients in the trivial A-module $\mathcal{K}$. Let us prove that the cohomological class of the cocycle ϑ in both cases is not trivial. If $\vartheta = d_{rsym}\omega$, $\omega \in C^1(N, \mathcal{K})$, then

$$\vartheta(e_i, e_j) = -(j+1)j\delta_{i+j,-1}, \text{ if } p = 0,$$

$$\vartheta(e_i, e_j) = -(-1)^i \delta_{i+j, p^m-1}, \text{ if } p > 0,$$

$$d_{rsym}\omega(e_i, e_j) = -(i+1)\omega(e_{i+j}), \text{ if } p = 0,$$

$$d_{rsym}\omega(e_i, e_j) = -\binom{i+j+1}{i})\omega(e_{i+j}), \;\; \text{if } p > 0.$$

In the case $p = 0$ we have a contradiction:

$$-2 = \vartheta(e_{-2}, e_1) = d_{rsym}(e_{-2}, e_1) = -\omega(e_{-1}),$$

$$-6 = \vartheta(e_{-3}, e_2) = d_{rsym}(e_{-3}, e_2) = -2\omega(e_{-1}).$$

Since, $d_{rsym}\omega(e_i, e_{p^m-i-1}) = 0$, we also obtain a contradiction even if $p > 0$.

Theorem 5.6. *Let $A = W_1^{rsym}$ if $p = 0$ and $A = W_1(m)$ if $p > 0$. Then the second right-symmetric cohomology space $H^2_{rsym}(A, \mathcal{K})$ has dimension 1 and is spanned by the class of the cocycle:*

$$\vartheta(e_i, e_j) = -(j+1)j\delta_{i+j,-1}, \; \textit{if } \; p = 0,$$

$$\vartheta(e_i, e_j) = -(-1)^i \delta_{i+j, p^m-1}, \; \textit{if } \; p > 0.$$

If A is considered as a Novikov algebra, then any Novikov central extension is split: $H^2_{nov}(W_1, \mathcal{K}) = 0$.

Recall that the Novikov cohomology is defined in [1].

Proof. For $u \in U = \mathcal{K}[[x^{\pm 1}]]$ let $\pi(u)$ be the coefficient of x^{-1}. Then $\pi(\partial(u)) = 0, \quad \forall u \in U$. Recall that $e_i = x^{i+1},\ i \in \mathbf{Z}$, and the multiplication in N is given by $a \circ b = \partial(a)b,\ a, b \in U$.

We prove that there is an isomorphism of A^{lie}-modules:

$$C^1(A, \mathcal{K}) \cong U_1. \tag{12}$$

A bilinear map

$$(\ ,\ ) : U_0 \times U_1 \to \mathcal{K}, \quad (u, v) \mapsto \pi(u \cdot v),$$

is compatible with the action of A^{lie} :

$$((a)_0(u), v) + (u, (a)_1(v))$$

$$= \pi(-(u \circ a) \cdot v - u \cdot (v \circ a) - u \cdot (a \circ v))$$

$$= \pi(-\partial(a \cdot (u \cdot v))) = 0,$$

for all $a \in A, u \in U_0, v \in U_1$. So, we have a pairing of A^{lie}-modules $(\ ,\ ) : U_0 \times U_1 \to \mathcal{K}$. This pairing is nondegenerate. Thus, the dual A^{lie}-module of U_0 is U_1. Since

$$(f \circ a)(b) = d_{rsym} f(b, a) = -f(b \circ a), \quad f \in C^1(N, k), a, b \in N,$$

we see that the A^{lie}-module $C^1(A, \mathcal{K})$ is isomorphic to the dual of U_0. This completes the proof of (12).

By Theorem 3.4,

$$H^2_{rsym}(A, \mathcal{K}) \cong H^1_{lie}(A, C^1(A, \mathcal{K})) \cong H^1_{lie}(W_1, U_1).$$

By the results of Gelfand and Fuks [11], the space $H^1_{lie}(W_1, U_1)$ is 1-dimensional and is spanned by the class of a cocycle $a \mapsto \partial^2(a)$. An analogous statement is also true in the case $p > 0$ [7]. The respective right-symmetric cocycle is the cocycle ϑ.

If $\psi : A \times A \to \mathcal{K}$ is a cocycle for a central extension in the category of Novikov algebras, then

$$\psi(a, b \circ c) - \psi(b, a \circ c) = 0, \quad \forall a, b, c \in A.$$

The algebra $A = W_1^{nov}$ has an element e_0 that has the property $e_0 \circ c = c$ for any $c \in A$. Take $a := e_0$. We have

$$\psi(e_0, b \circ c) = \psi(b, e_0 \circ c) = \psi(b, c), \quad \forall b, c \in A.$$

Therefore, for $\omega \in C^1_{nov}(A, \mathcal{K}) = C^1(A, \mathcal{K})$, such that $\omega(a) = -\psi(e_0, a)$, we have

$$\psi(b, c) = -\omega(b \circ c) = d_{nov}\omega(b, c).$$

Recall that $d_{nov}\phi = d_{right}\phi$ for any $\phi \in C^1(A, \mathcal{K})$. So, any 2-cocycle of W_1^{nov} (in the sense of Novikov) with coefficients in the trivial module is a coboundary. □

5.5. Cohomology of W_n^{rsym} in an antisymmetric module

Recall that a multiplication in W_n^{rsym} is given by $a\partial_i \circ b\partial_j = b\partial_j(a)\partial_i$, where $a, b \in U = \mathcal{K}[[x^{\pm 1}, \ldots, x^{\pm 1}]]$. We endow U with the structure of an antisymmetric W_n^{rsym}-module: $u \circ a\partial_i = a\partial_i(u)$.

Let $\Omega_n = \{u\, dx_1 \wedge \cdots \wedge dx_n : u \in U\}$ be an antisymmetric W_n^{rsym}-module of n-dimensional differential forms:

$$(u\, dx_1 \wedge \cdots dx_n) \circ a\partial_i = \partial_i(au)\, dx_1 \wedge \cdots \wedge dx_n.$$

Let M be a W_n^{rsym}-module. We construct a cup product of W_n^{rsym}-modules:

$$U \times \Omega^n \otimes M \to \bar{M}, \quad (u \cup v\, dx_1 \wedge \cdots \wedge dx_n \otimes m) = \pi(uv)m, \tag{13}$$

where $\pi(u)$ for $u \in U$ denotes a coefficient of u at $x_1^{-1} \dots x_n^{-1}$. Recall that $\bar{M}$ is an antisymmetric A-module corresponding to M, such that $\bar{r}_a = r_a - l_a, \bar{l}_a = 0$. Note that

$$\pi(der_i(u)) = 0, \ \ \forall u \in U, i = 1, \dots, n.$$

Therefore,

$$\begin{aligned}(u \circ a\partial_i) \cup (v\, dx_1 \wedge \cdots dx_n \otimes m) + u \cup [v\, dx_1 \wedge \cdots dx_n \otimes m, a\partial_i] \\ = a\partial_i(u) \cup (v\, dx_1 \wedge \cdots \wedge dx_n \otimes m) + u \cup (\partial_i(av)\, dx_1 \wedge \cdots \wedge dx_n \otimes m) \\ + u \cup v\, dx_1 \wedge \cdots \wedge dx_n \otimes [m, a\partial_i] \\ = \pi(a\partial_i(u)v + u\partial_i(av))\, m + \pi(uv)\, [m, a\partial_i] \\ = \pi(\partial_i(auv)) + \pi(uv)\, [m, a\partial_i] \\ = (u \cup v\, dx_1 \wedge \cdots \wedge dx_n) \otimes [m, a\partial_i] \\ = (u \cup v\, dx_1 \wedge \cdots \wedge dx_n \otimes m) \circ a\partial_i.\end{aligned}$$

and the definition of the cup product (13) is correct.

Theorem 5.7. *Let M be an antisymmetric W_n^{rsym}-module. The cup product of W_n^{rsym}-modules (13) induces an isomorphism*

$$H^{k+1}_{rsym}(W_n, M) \cong Z^1_{rsym}(W_n, U) \otimes H^k_{lie}(W_n, \Omega^n \otimes M), \ \ k > 0.$$

Then we have an isomorphism

$$Z^1_{rsym}(W_n, U) \cong B^1_{rsym}(W_n, U) = \{dx_i : i = 1, \dots, n\} \cong \wedge^1.$$

Proof. We will argue as in the previous subsection. For a W_n-module M, its dual module is denoted by M'. Consider $\wedge^1 = \{dx_1, \dots, dx_n\}$ as a trivial module over the Lie algebra W_n. Endow $U \otimes \wedge^1$ with the structure of a W_n-module using the natural W_n-module structure on $U = \mathcal{K}[[x_1, \dots, x_n]]$ and the trivial W_n-module structure on $\wedge^1$:

$$(u \otimes dx_i)b\partial_j = b\partial_j(u) \otimes dx_i.$$

It is easy to see that

$$C^1(W_n, \mathcal{K}) \cong \wedge^1 \otimes U',$$

since for any $f \in C^1(W_n, \mathcal{K})$,

$$[f, a\partial_i](b\partial_j) = (f \circ a\partial_i)(b\partial_j) = -f(a\partial_i(b)\partial_j).$$

The bilinear map (13) is nondegenerate and gives a pairing of modules over a Lie algebra W_n. So,

$$U' \cong \Omega_n.$$

Therefore, there exists an isomorphism of W_n-modules:

$$C^1(W_n, \mathcal{K}) \cong \wedge^1 \otimes \Omega_n, \ \ C^1(W_n, M) \cong \wedge^1 \otimes \Omega_n \otimes M,$$

and by Theorem 3.4

$$H^{k+1}_{rsym}(W_n, M) \cong \wedge^1 \otimes H^k_{lie}(W_n, \Omega^n \otimes M), \ \ k > 0. \quad \square$$

Corollary 5.8. $H^{k+1}_{rsym}(W_n, \mathcal{K}) \cong \wedge^1 \otimes H^k_{lie}(W_n, \Omega^n), \ \ k > 0.$

Recall that $H^*_{lie}(W_n, \Omega^*)$ is determined by Gelfand and Fuks [11].

5.6. Cohomology of gl_n^{rsym} in an antisymmetric module

Theorem 5.9. *Let $A = gl_n^{rsym}$, char $k = 0$. Let M be a finite-dimensional A-module such that, as an A^{lie}-module, M is a tensor module. Then the cup product $M \times \mathcal{K} \to M$, $m \cup \lambda = \lambda m$, induces an isomorphism*

$$H^{k+1}_{rsym}(A, M) \cong Z^1_{rsym}(A, M) \otimes H^k_{lie}(A, \mathcal{K}), \quad k > 0.$$

Proof. By Theorem 3.4,

$$H^{k+1}_{rsym}(A, M) \cong H^k_{lie}(A, C^1(A, M)), \quad k > 0.$$

By Theorem 2.1.2 of [11],

$$H^k_{lie}(A, C^1(A, M)) \cong H^k_{lie}(A, \mathcal{K}) \otimes C^1(A, M)^{A^{lie}}.$$

It remains to note that

$$C^1(A, M)^{A^{lie}} := \{f \in C^1(A, M) : [f, a] = 0, \ \forall a \in A\}$$

is exactly $Z^1_{rsym}(A, M)$. This is obvious:

$$[f, a](b) = (f \circ a)(b) = d_{rsym} f(b, a), \quad \forall a, b \in A. \quad \square$$

Corollary 5.10. *Let $A = gl_n^{rsym}$ and M be an irreducible antisymmetric A-module. Then $H^k_{rsym}(A, M) \neq 0, k \geq 0$, if and only if $M = A$ and*

$$H^{k+1}_{rsym}(gl_n, (gl_n)_{anti}) \cong H^k_{lie}(gl_n, \mathcal{K}), k \geq 0.$$

In particular, $H^k_{rsym}(gl_n, \mathcal{K}) = 0, \ \ k > 0$.

Proof. Since M is antisymmetric,

$$Z^1_{rsym}(A, M) = \{f : A \to M : f(b \circ a) = f(b) \circ a\}.$$

Thus, any $f \in Z^1_{rsym}(A, M)$ gives us a homomorphism of right modules $f : A \to M$. Since right modules M and A are irreducible, by Schur's lemma $Z^1_{rsym}(A, M) \cong \mathcal{K}$ if $M \cong A$, and $Z^1_{rsym}(A, M) = 0$ if $M \not\cong A$. $\square$

5.7. Cohomology of gl_n^{rsym} in a regular module

Theorem 5.11. *Let $A = gl_n^{rsym}$ over a field $\mathcal{K}$ of characteristic 0 and $M = A$ be its regular module. Then the cup product $M \times \mathcal{K} \to M, m \cup \lambda = \lambda m$, induces an isomorphism*

$$H^{k+1}_{rsym}(A, M) \cong Z^1_{rsym}(A, A) \otimes H^k_{lie}(gl_n, \mathcal{K}), \ Z^1_{rsym}(A, A) \cong sl_n, \quad k \geq 0.$$

In particular, any cocycle class in $H^{k+1}_{rsym}(gl_n, gl_n)$ has a representative that can be written as $ad\, X \cup \psi$, where $\psi \in Z^k_{lie}(gl_n, \mathcal{K})$.

Proof. Any right-symmetric 1-cocycle of an associative algebra A is also an associative 1-cocycle and conversely, any associative 1-cocycle is a right-symmetric 1-cocycle. So, $Z^1_{rsym}(gl_n, gl_n) \cong Z^1_{ass}(gl_n, gl_n)$. Any derivation of the associative algebra gl_n is a derivation of the Lie algebra of gl_n. Any derivation of gl_n^{lie}, except $a \mapsto tr\, a$, is inner. So, the following sequence is exact:

$$0 \to Z^1_{rsym}(gl_n, gl_n) \to Z^1_{lie}(gl_n, gl_n) \to \mathcal{K} \to 0.$$

In particular,

$$Z^1_{rsym}(gl_n, gl_n) = \{ad\, X : X \in sl_n\} \cong Z^1_{lie}(sl_n, sl_n) \cong sl_n.$$

It remains to use Theorem 3.4. $\square$

Corollary 5.12. *An algebra gl_n as a right-symmetric algebra has nontrivial deformations. Any right-symmetric local deformation (2-cocycle of the regular module) is equivalent to a 2-cocycle η_X of the form*

$$\eta_X(a,b) = (tr\, b)[X,a],$$

for some $X \in sl_n$. Any local right-symmetric deformation can be extended. Any formal right-symmetric deformation of gl_n is equivalent to a deformation of the form

$$\mu_t(a,b) = a \circ b + t\,(tr\, b)[X,a], \quad X \in sl_n,$$

where $(a,b) \mapsto a \circ b$ is the usual associative multiplication of matrices.

Proof. These statements can be obtained from the following cohomological facts:

$$H^2_{rsym}(gl_n, gl_n) \cong Z^1_{right}(gl_n, gl_n) \otimes H^1_{lie}(gl_n, \mathcal{K}) \cong \{ad\, X \cup tr : X \in sl_n\},$$

$$H^3_{rsym}(gl_n, gl_n) = 0,$$

and Corollary 4.2. □

Remark. For $\omega \in C^1(gl_n, gl_n)$, $\omega(a) = (tr\, a)X$, we have

$$(\eta + d_{rsym}\omega)(a,b) = (tr\, b)[X,a] + (tr\, b)a \circ X - (tr\, a \circ b)X + (tr\, a)X \circ b = \bar{\eta}_X(a,b),$$

where

$$\bar{\eta}_X(a,b) = (tr\, b)X \circ a - (tr\, a \circ b)X + (tr\, a)X \circ b.$$

Therefore, $[\eta_X] \sim [\bar{\eta}_X]$. Note that $\bar{\eta}_X$ is a symmetric cocycle:

$$\bar{\eta}(a,b) = \bar{\eta}(b,a).$$

The extension formula for $\bar{\eta}_X$ is a little bit complicated. It can be given by

$$\bar{\mu}_t(a,b) = \Phi_t^{-1}(\mu_t(\Phi_t(a), \Phi_t(b))),$$

where

$$\Phi_t = id + t\,\omega.$$

In particular,

$$\Phi_t^{-1}(a) = a - t\,(tr\, a)X + t^2\,(tr\, a)^2 X^2 - \cdots,$$

and some of the beginning terms of $\bar{\mu}_t$ have the form

$$\bar{\mu}_t(a,b) = a \circ b + t\,X \circ ((tr\, a)b - (tr\, a \circ b) + (tr\, b)a)$$

$$+t^2\,(tr\, a\, tr\, b - (tr\, a \circ b)^2 X^2 - (tr\, a\, tr\,(X \circ b))X - (tr\,(a \circ X)\, tr\, b)X) + \cdots.$$

Thus, the right-symmetric extension can be constructed in a such way that the corresponding Lie multiplication will not be changed:

$$\bar{\mu}_t(a,b) - \bar{\mu}_t(b,a) = [a,b].$$

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