C. R. Acad. Sci. Paris, t. 332, Série I, p. 1047–1052, 2001
Algèbres de Lie/*Lie Algebras*

Leibniz algebras in characteristic p

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(Reçu le 7 septembre 2000, accepté après révision le 5 mars 2001)

Abstract. We give the definition of restrictness for Leibniz algebras in characteristic p. We prove that the cohomology of Leibniz algebras with coefficients in an irreducible module is trivial, if the module is not restricted. The number of irreducible antisymmetric modules with nontrivial cohomology is finite. A Leibniz algebra is called simple, if it has no proper ideal except ideal generated by squares of its elements. We describe simple Leibniz algebras with Lie factor isomorphic to $\mathfrak{sl}_2$ and p^m-dimensional Zassenhaus algebra $W_1(m)$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Algèbres de Leibniz en caractéristique p

Résumé. *Nous définissons les algèbres de Leibniz restreintes en caractéristique* p *par analogie avec le cas des algèbres de Lie. On montre que la cohomologie des algèbres de Leibniz restreintes, à coefficients dans un module irréductible, est triviale si le module n'est pas restreint. Le nombre de modules irréductibles antisymétriques avec cohomologie non triviale est fini. Une algèbre de Leibniz est dite simple si elle n'a qu'un idéal propre engendré par les carrés de ses éléments. Nous donnons la description des algèbres de Leibniz simples avec le facteur de Lie isomorphe à l'algèbre de Lie* $\mathfrak{sl}_2$ *et l'algèbre de Zassenhaus* $W_1(m)$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Une algèbre à identité $[[x,y],z]=[x,[y,z]]-[y,[x,z]]$ est appelée *algèbre de Leibniz*. Nous définissons les algèbres de Leibniz restreintes en caractéristique p par analogie avec le cas des algèbres de Lie.

THÉORÈME 1. – *Le cohomologie des algèbres de Leibniz restreintes à coefficients dans le module irréductible est, triviale si le module n'est pas restreint. Le nombre de modules irréductibles antisymétriques avec cohomologie non triviale est fini.*

Une algèbre de Leibniz est dite *simple* si tout ideal propre est engendré par les carrés. Nous donnons la description des algèbres de Leibniz simples avec le facteur de Lie isomorphe à l'algèbre de Lie $\mathfrak{sl}_2$ et

Note présentée par Jean-Louis KOSZUL.

S0764-4442(01)01941-3/FLA

l'algèbre de Zassenhaus

$$W_1(m)=\left\{e_i:[e_i,e_j]=\left(\binom{i+j+1}{j}-\binom{i+j+1}{i}\right)e_{i+j},\ -1\leqslant i,j\leqslant p^m-2\right\}.$$

THÉORÈME 2 ($p>3$). – *Chaque algèbre de Leibniz non standard simple avec le facteur de Lie $\mathfrak{sl}_2$ est isomorphe à l'algèbre de dimension p* :

$$\mathfrak{sl}_2+V_{p-4}=\left\{e:=x\partial_y,\ e_0:=-x\partial_x+y\partial_y,\ e_+:=-y\partial_x\right\}+\left\{x^iy^j:i+j=p-4,\ 0\leqslant i,j\leqslant p-4\right\}$$

avec multiplication

$$[e_-,e_+]=e_0,\quad [e_0,e_+]=2e_0,\quad [e_0,e_-]=-2e_-,\quad [e_-,e_-]=\alpha y^{p-4},\quad [e_+,e_+]=\beta x^{p-4},$$

pour certains $\alpha,\beta\in k$.

Soit U_t un $W_1(m)$-module antisymétrique défini sur l'algèbre $U=O_1(m)$ par la règle

$$(u\partial)_t v=a\partial(v)+t\partial(u)v.$$

THÉORÈME 3 ($p>7$). – *Chaque algèbre de Leibniz simple non standard avec le facteur de Lie isomorphe à $W_1(m)$ est isomorphe à l'une des extensions de Leibniz non scindées de $W_1(m)$ par un module irréductible antisymétrique, suivantes* :

M	*Principal cocycles*	dim
U_{-2}	$\partial^{p^\ell}(a)\partial^{p^k}(b)+\partial^{p^\ell}(a)\partial^{p^k}(b)-\partial^{p^k+p^\ell}(a)b,\ 0<k\leqslant\ell<m,$ $\partial^{p^k}(a)b,\ 0<k<m$	$\binom{m+1}{2}-1$
U_{-1}	$\partial(a)b$, $x^{(p^m-1)}ab$	2
k	$\pi_{p^m-1}(a\partial^3(b))$	1
$\overline{U}_1$	$\partial^{p^k+3}(a)b-\partial^3(b)\partial^{p^k}(a)+2\partial^3(a)\partial^{p^k}(b),\ 0<k<m$	$m-1$
U_2	$-(1/2)\partial^4(a)b+\partial^2(a)\partial^2(b)$	1
U_3	$-(3/10)\partial^5(a)b-(1/2)\partial^4(a)\partial(b)+\partial^3(a)\partial^2(b)+\partial^2(a)\partial^3(b)$	1

Let k be an algebraically closed field of characteristic $p>0$. An algebra Q is called (left) Leibniz [4], if

$$\big[[x,y],z\big]=\big[x,[y,z]\big]-\big[y,[x,z]\big],\quad \text{for any } x,y,z\in Q.$$

In general, Leibniz algebras are not simple:

$$Q^{\mathrm{ann}}=\left\{[x,y]+[y,x],\ x,y\in Q\right\}$$

is an Abelian ideal of Q, such that $[Q^{\mathrm{ann}},Q]=0$. We will say that Q *has a Lie-factor* $L=Q^{\mathrm{lie}}=Q/Q^{\mathrm{ann}}$. The following notion of simplicity for Leibniz algebras is given in [2].

DEFINITION 1. – An algebra Q is *a simple Leibniz* algebra, if all its ideals are $0, Q^{\mathrm{ann}}, Q$.

If the Leibniz algebra is simple, then its Lie-factor is simple and its annihilator as a module over Lie-factor is irreducible.

Example 1. – Any simple Lie algebra is simple as a Leibniz algebra.

Let Q be a Leibniz algebra and M be a Q-module. Recall that the Q-module M is called *symmetric*, if $[x,m]+[m,x]=0$, and *antisymmetric*, if $[m,x]=0$, for any $x\in Q, m\in M$.

DEFINITION 2. – Let L be a Lie algebra and M be an antisymmetric L-module. Call M *strongly antisymmetric*, if $LM=M$, i.e., $M=\{(x)_M m : x\in L,\ m\in M\}$.

Example 2. – Any nontrivial irreducible module of a Lie algebra is strongly antisymmetric. A trivial module is not strong.

PROPOSITION 1. – *Let L be a Lie algebra and M be a strongly antisymmetric L-module. Then there exists a Leibniz algebra Q with Lie-factor L and annihilator M.*

Proof. – Take Q as a semidirect product of L by the antisymmetric module M. □

Leibniz algebras with given Lie-factor L and annihilator M are ruled by the second Leibniz cohomology group $\mathrm{HL}^2(L,M)$, where M is considered as antisymmetric L-module [4]. Concerning simple Leibniz algebras this means the following. The classification problem for simple Lie algebras with given Lie-factor L is equivalent to the problem of describing nonsplit extensions of adjoint L-modules by irreducible modules [2]. Simple Leibniz algebras $L+M$, with multiplication $[x+m,y+n]=[x,y]+[x,n]$, where M is irreducible antisymmetric L-module, we call as *standard* simple Leibniz algebras with Lie-factor L. In particular, a simple Lie algebra L is standard simple.

Let ℓ_x, r_x be the left-multiplication and right-multiplication operators $\ell_x : Q\to Q$, $y\mapsto [x,y]$, $r_x : Q\to Q$, $y\mapsto [y,x]$. Let $U(Q)$ be the universal enveloping algebra of Q.

DEFINITION 3. – A Leibniz algebra Q over a field of characteristic $p>0$ is called *weakly restricted*, if for any $x\in Q$, there exists some nonnegative integer $\ell(x)$ and some element of Q, denoted by $x^{[p^{\ell(x)}]}$, such that $\ell_x^{p^{\ell(x)}}=\ell_{x^{[p^{\ell(x)}]}}$. If $\ell(x)=1$ for any x, i.e., for any $x\in Q$, there exists some $x^{[p]}\in Q$, such that $\ell_x^p=\ell_{x^{[p]}}$, then Q is called *restricted* or *p-Leibniz algebra*. In this case a map $Q\to Q$, $x\mapsto x^{[p]}$ is called a p-map.

Recall that any module M of the Leibniz algebra Q can be considered as a left module of the universal enveloping algebra $U(Q)$, by the following rules $\ell_x m=[x,m]$, $r_x m=[m,x]$ (*see* [4]).

DEFINITION 4. – Let Q be a weakly restricted Leibniz algebra. The Q-module M is called *weakly restricted*, if $\ell_x^{p^{\ell(x)}}m=\ell_{x^{[p^{\ell(x)}]}}m$, for any $x\in Q$, $m\in M$.

Example 3. – Any restricted Lie algebra is restricted as a Leibniz algebra.

LEMMA 2. – *Let Q be a weakly restricted Leibniz algebra. Then for any $x\in Q$, the element $z_x=\ell_x^{p^{\ell(x)}}-\ell_{x^{[p^{\ell(x)}]}}\in U(Q)$ is central*: $[z_x,\ell_y]=0$, $[z_x,r_y]=0$, *for any $y\in Q$.*

THEOREM 3. – *Let Q be a weakly restricted Leibniz algebra of characteristic $p>0$ and M be irreducible Q-module, such that $\mathrm{HL}^*(Q,M)\neq 0$. Then M is weakly restricted.*

Proof. – Let $\rho : Q\to \mathrm{CL}^*(Q,M)$ be representation of Q on the Leibniz cochain complex $\mathrm{CL}^*(Q,M)$ and $i(x) : \mathrm{CL}^*(Q,M)\to \mathrm{CL}^{*-1}(Q,M)$ be interior product endomorphism. Then for any $x\in Q$, [4] the following relations hold

$$\mathrm{d}\, i(x)+i(x)\,\mathrm{d}=\rho(x),\quad \mathrm{d}\,\rho(x)=\rho(x)\,\mathrm{d}. \tag{1}$$

By Lemma 1 we have,

$$\rho\big(z(x)\big)\psi(x_1,\ldots,x_k) = \big(z(x)\big)_M\big(\psi(x_1,\ldots,x_k)\big), \quad \psi \in \mathrm{CL}^k(Q,M).$$

From (1) it follows that

$$\rho(x)^{p^{\ell(x)}} = \rho(x)^{p^{\ell(x)}-1}\big(\mathrm{d}\, i(x) + i(x)\,\mathrm{d}\big) = \mathrm{d}\,\rho(x)^{p^{\ell(x)}-1} i(x) + \rho(x)^{p^{\ell(x)}-1} i(x)\,\mathrm{d}.$$

Therefore,

$$\big(z(x)\big)_M\psi = \mathrm{d}\,\Gamma\psi + \Gamma\,\mathrm{d}\psi,$$

where

$$\Gamma = \rho(x)^{p^{\ell(x)}-1} i(x) - i\big(x^{[p^{\ell(x)}]}\big).$$

By the Schur lemma $(z(x))_M$ is invertible for some $x \in Q$, if M is not weakly restricted. That ends the proof. □

Remark 1. – A similar result for modular Lie algebras was proved in [1].

COROLLARY 4. – *Let M be an irreducible antisymmetric module of the Lie algebra L in characteristic $p > 0$. Then the number of nonisomorphic antisymmetric modules M with property $\mathrm{HL}^*(L,M) \neq 0$ is finite. In particular, the number* (*up to isomorphism*) *of non-standard simple Leibniz algebras with given Lie-factor L is finite.*

Proof. – From results of [3] it follows that the number of weakly restricted nonisomorphic modules is finite.

Let $\mathfrak{sl}_2 = \{e_-, e_0, e_+ : [e_-, e_+] = e_0,\ [e_0, e_+] = 2e_0,\ [e_0, e_-] = -2e_-\}$ be three-dimensional simple Lie algebra over a field k of characteristic $p > 3$. Consider the representation of $\mathfrak{sl}_2$ in $k[x,y]$ given by $e_- \mapsto x\partial_y$, $e_0 \mapsto -x\partial_x + y\partial_y$, $e_+ \mapsto -y\partial_x$. Then

$$V_q = \big\{x^i y^j : i + j = q,\ 0 \leqslant i,j \leqslant q\big\}, \quad 0 < q < p,$$

has a structure of irreducible $\mathfrak{sl}_2$-module of dimension $q+1$. It has the highest vector $v = y^q$ and highest weight q under Borel subalgebra $\langle e_0, e_+\rangle$: $e_+v = 0$, $e_0v = qv$. Any irreducible p-module of $\mathfrak{sl}_2$ is isomorphic to V_q, for some $0 < q < p$, [5]. □

THEOREM 5 ($p > 3$). – *Any non-standard simple Leibniz algebra with Lie-factor $\mathfrak{sl}_2$ is isomorphic to a p-dimensional non-split Leibniz extension of $\mathfrak{sl}_2$ by a $(p-3)$-dimensional antisymmetric irreducible module V_{p-4} with cocycle $(e_-, e_-) \mapsto \alpha y^{p-4}$, $(e_+, e_+) \mapsto \beta x^{p-4}$, $(e_i, e_j) \mapsto 0$, $(i,j) \neq (-,-), (+,+)$, where $\alpha, \beta \in k$.*

Let

$$O_1(m) = \left\{x^{(i)} : x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)},\ 0 \leqslant i,j < p^m\right\}$$

be the divided power algebra. The Zassenhaus algebra $L = W_1(m)$ can be defined as a Lie algebra of special derivations of $U = O_1(m)$. One can get a basis $e_i := x^{(i+1)}\partial$, $-1 \leqslant i \leqslant p^m - 2$. Then

$$[e_i, e_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) e_{i+j}.$$

The p^m-dimensional algebra $L = W_1(m)$ is simple and weakly restricted. It is restricted if and only if $m = 1$. It has grading

$$L = \bigoplus_{i=-1}^{p^m-2} L_i, \quad [L_i, L_j] \subseteq L_{i+j},$$

and filtration

$$\mathcal{L}_{-1} = L \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_{p^m-2} \supset 0,$$
$$\mathcal{L}_i = \bigoplus_{j \geqslant i} L_j, \quad \mathcal{L}_i \mathcal{L}_j \subseteq \mathcal{L}_{i+j}.$$

We see that $\mathcal{L}_0$ is a maximal subalgebra of $W_1(m)$ and $\mathcal{L}_{i+1}$ is a subalgebra of $\mathcal{L}_i$ with codimension 1. The $W_1(m)$-module U_t is defined on the divided power vector space $O_1(m)$ by the following rule

$$(e_i)_t x^{(j)} = \left(\binom{i+j}{i+1} + t \binom{i+j}{i} \right) x^{(i+j)},$$

where $t \in k$. These modules are irreducible, if $t \neq 0, 1$ and restricted if $t \in \mathbb{Z}/p\mathbb{Z}$. As far as the cases $t = 0, 1$, the module U_0 has one-dimensional submodule $\langle 1 \rangle$ and U_1 has $p^m - 1$-dimensional submodule $\overline{U}_1 := \{x^{(i)} : 0 \leqslant i \leqslant p^m - 2\}$. There is an isomorphism of modules $\overline{U}_1 \cong U_0/\langle 1 \rangle$. The modules U_t, $t \neq 0, 1$, $t \in \mathbb{Z}/p\mathbb{Z}$ and $\overline{U}_1$, k are restricted and any restricted irreducible $W_1(m)$-module is isomorphic to one of these modules.

THEOREM 6. – *Let $p > 7$ and M be an irreducible antisymmetric $W_1(m)$-module. If the second Leibniz cohomology group $\mathrm{HL}^2(W_1(m), M)$ is nonzero, then M is isomorphic to one of the following six modules U_{-2}, U_{-1}, k, $\overline{U}_1$, U_2, U_3. Any non-standard simple Leibniz algebra is isomorphic to one of the corresponding nonsplit Leibniz extensions of $W_1(m)$ by M. Basic cocycles $(a, b) \mapsto \psi(a, b)$ and dimensions of $\mathrm{HL}^2(W_1(m), M)$ are given in the following table.*

M	ψ	dim
U_{-2}	$\partial^{p^\ell}(a)\partial^{p^k}(b) + \partial^{p^\ell}(a)\partial^{p^k}(b) - \partial^{p^k+p^\ell}(a)b$, $0 < k \leqslant \ell < m$,	$\binom{m+1}{2} - 1$
	$\partial^{p^k}(a)b$, $0 < k < m$	
U_{-1}	$\partial(a)b,\ x^{(p^m-1)}ab$	2
k	*Coefficient at* $x^{(p^m-1)}$ of $a\partial^3(b)$ (*Virasoro cocycle*)	1
$\overline{U_1}$	$-\partial^{p^k+3}(a)b - 3\partial^{p^k+2}(a)\partial(b) + 3\partial^{p^k+1}(a)\partial^2(b)$	$m - 1$
	$-3\partial^2(a)\partial^{p^k+1}(b) + \partial^3(a)\partial^{p^k}(b) + \partial^{p^k}(a)\partial^3(b)$, $0 < k < m$	
U_2	$-(1/2)\partial^4(a)b + \partial^2(a)\partial^2(b)$	1
U_3	$-(3/10)\partial^5(a)b - (1/2)\partial^4(a)\partial(b) + \partial^3(a)\partial^2(b) + \partial^2(a)\partial^3(b)$	1

Remark 2. – From our results it is easy to obtain similar description for simple Leibniz algebras with Lie-factor isomorphic to Lie algebra of formal vector fields on the circle in characteristic 0.

Acknowledgements. The authors are deeply grateful to the INTAS foundation and the Royal Society of Sweden for support.

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